MATH 351 Fall 2015 Homework 12 Due: Tuesday 12/8

Read Chapter 3 section 3 and Chapter 4 sections 1,2,3. For each of the following problems, use the metric space $(\mathbb{R}, d)$ where d is the usual metric (unless otherwise stated).

(1) Suppose that (X, d) be a metric space.
  (a) Define, in your own words, what it means for (X, d) to be a compact metric space.
  (b) Give an example of (X, d) that is complete, but not compact.
  (c) Show that if (X, d) is a compact metric space, then X is complete.

(2) Decide whether each of the following is compact or not.
  (a) $\mathbb{Q}$
  (b) $\mathbb{Q} \cap [0, 1]$
  (c) $\mathbb{R}$
  (d) $\mathbb{Z}$
  (e) $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$

(3) For each of the sets above that are not compact, find an open cover that does not have a finite subcover.

(4) Show that the metric space $(\mathbb{R}, d)$, where $d((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|, |y_2 - y_1|\}$ is a complete metric space.

(5) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ so that $f^{-1}(\{a\})$ is not closed.

(6) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ and two real numbers a, b so that $f^{-1}((a, b))$ is not open.

(7) Prove or disprove:

$$\lim_{x \to 0} \sin(1/x)$$

exists.

(8) Suppose that $f : [a, b] \to [0, \infty)$ and suppose that for every $\varepsilon > 0$, $\{x \in [a, b] | f(x) \geq \varepsilon\}$ is finite. Show that for every $c \in [a, b]$,

$$\lim_{x \to c} f(x) = 0.$$

(MGC #4) Given a set $Y \subset \mathbb{R}$ and function $f : \mathbb{R} \to \mathbb{R}$,
  (a) Define the set $f^{-1}(Y)$
  (b) Show that f is continuous at every $a \in \mathbb{R}$ if and only if $f^{-1}(A)$ is open for all open sets $A \subset \mathbb{R}$.

(JFF) Give an example of a metric space in which the Heine-Borel theorem does not hold.