MATH 351 Fall 2015 Homework 11 Due: Tuesday 12/1

Read Chapter 2 sections 7 and 8 and Chapter 3 sections 1,2. For each of the following problems, use the metric space $(\mathbb{R}, d)$ where d is the usual metric (unless otherwise stated).

(1) For each of the following sets, determine whether it is open, closed, or neither. Prove your answer.
  (a) $(0, 1)$
  (b) $[0, 1]$
  (c) $(0, 1]$

(2) Prove Theorem 2.7.7 in your book.

(3) Here I want you to prove a theorem that I've broken into several smaller parts to make it more approachable for you.
Consider the series $\sum_{n=1}^{\infty} a_n$ with $a_n \neq 0$ for all $n \in \mathbb{N}$. Assume that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = r < 1.$$

  (a) Let $\tilde{r}$ be a real number satisfying $r < \tilde{r} < 1$. How do we know such an $\tilde{r}$ exists?
  (b) Explain why we know that $\exists N \in \mathbb{N}$ so that $|a_n|\tilde{r} \geq |a_{n+1}|$ for all $n \geq N$.
  (c) How do we know that $|a_N| \sum_{n=1}^{\infty} \tilde{r}^n$ converges?
  (d) Show that $\sum_{n=1}^{\infty} a_n$ converges.
  (e) What did we call this rule in Calculus 2?

(4) Prove or disprove: There exists a closed set A so that $\mathbb{Q} \subseteq A$.

(5) Prove or disprove: The set of all isolated points of a set A is closed.

(6) Prove that x is an isolated point of a set $X \subset \mathbb{R}$ if and only if $\exists \varepsilon > 0$ so that $\mathcal{N}_\varepsilon(x) \cap X = \{x\}$.

(7) Show that the Cantor set is compact.