MATH 351 – FOM HOMEWORK

1. Problems

Here are some FOM problems. If you try them and have any trouble, ask me about them. The problems will help in creating techniques for doing Analysis proofs..

(1) Prove or disprove:
   (a) $(1,3)$ is a subset of $(2,4)$.
   (b) $(2,3)$ is a subset of $(1,4)$.

(2) Use problem 1 to make a general statement about a, b, c, and d so that

$$(a,b) \subset (c,d).$$

(3) Prove or disprove the statement "If a is a rational number and b is an irrational number, then $a+b$ is irrational."

(4) Consider the **Well-Ordering Principle:**

Any nonempty subset $S \subset \mathbb{N}$ contains a least element.

   (a) Write this in another way so that the meaning of "contains a least element" is more mathematical.
   (b) Write the Principle of Mathematical Induction.
   (c) Use the Well-Ordering Principle to prove the Principle of Mathematical Induction.
   (d) Use the Principle of Induction to prove the Well-Ordering Principle.

(5) Translate each statement into English.
   (a) $\forall\, p \in P (q > p)$. (Here, P is the set of prime numbers.)
   (b) $\forall\, v \in \mathbb{R}\ \ \exists\, u \in \mathbb{R}\ (\frac{1}{u} = v)$.
   (c) $\forall\, x \in \mathbb{R}\ \ \forall\, y \in \mathbb{R}\ (x^3 - x = y^3 - y \Rightarrow x = y)$. (Hint: This is really a statement about a particular function.)
   i.) Is the statement in part 5b True or False? Why?
   ii.) Is the statement in part 5c True or False? Why?

(6) Let $S \subseteq \mathbb{R}$. Consider the following two quantified statements, labelled F and G.

$$F : \exists z \in S\ \ \forall x \in S\ \ (x \leq z) \qquad\qquad G : \forall x \in S\ \ \exists z \in S\ \ (x \leq z)$$

   (a) Write down the negation of statement F without using the negation symbol.
   (b) Describe in words what statement G means.
   (c) Are the two statements logically equivalent? If you think they are say why. If you think they aren't, say why and give a set which makes one true and the other false.

(7) For this problem, we are considering the following statement:

If $A \subseteq B$ and $C \subseteq D$, then $A \cap C \subseteq B \cap D$.

   (a) If possible, draw a Venn Diagram that shows this statement.
   (b) Give an example of sets A, B, C, and D making this statement true.

(c) Below is a "proof" of this statement. Explain whether it does or does not prove the statement. If you think it does prove the statement, discuss what you would do to improve the proof. If you think it doesn't prove the statement, specifically state *all* the mistakes.

Proof: Assuming the hypothesis. Take $x \in A$. Then $x \in B$ because $A \subseteq B$. Take $y \in C$. Then $y \in D$ because $C \subseteq D$. So x and y are both in $B \cap D$. Thus $A \cap C \subseteq B \cap D$.

(8) Consider the statement

"If $x < y$ are elements of $\mathbb{Q}$, then there exists $r > 0$ in $\mathbb{R}$ such that $x + r < y - r$."

(a) Use quantifiers on the predicate $Q(x, y, r)$:

$$\text{“}x + r < y - r\text{”}$$

to create an equivalent statement.

(b) Prove or disprove this statement.

(9) Use a truth table to prove that an implication $P \Rightarrow Q$ is logically equivalent to $Q \vee \neg P$.