MATH 351 Fall 2015 Exam 2 Due: Tuesday 12/15 by 5pm

<u>**Directions:**</u> You may use your own textbook but not any solutions manual. You may use **your own** notes and **your own** homework. You may **not** use the internet or any other electronic or printed resource including calculators, mathematical software, or other books. You may not use anyone elses notes, you may not use any one else's homework. If you work on this exam in a public space you must keep your work private. Do not leave your work where it is visible. Erase boards if you work on them. You may not discuss this exam with or near any person except Dr. Heather Moon. Failure to follow the letter and spirit of these instructions will result in you failing the course. Your solutions should be **typeset with LaTeX**. Be sure to check that each of your proofs is clear, correct, and complete. Your audience is not Heather, nor is it Yingyi. You must provide sufficient reasoning to completely convince any of your classmates. Any significant results that you use must be cited. Problems from homework may be cited, but you must be able to prove them. **If the problem is taken directly from a homework or class assignment, you must re-prove it entirely.**

Do not turn in this sentence or the above directions with your work.

The exam is due <u>**no later than 5 PM on Tuesday, December 15**</u>.

(1) Given the set
$$A = \left\{ 1 - \frac{n-1}{n+3} \,\middle|\, n \in \mathbb{N} \right\}.$$
Find $\inf A$ and then prove your result.

(2) Given the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$.

(a) Let $\mathcal{C} \subset \mathbb{R}$. Clearly, write out $f(\mathcal{C})$, for the above f.

(b) Using your previous answer, what is $f([2,3])$?

(c) **Using the definition** of closed set, show that $f([2,3])$ is closed in $\mathbb{R}$.

(d) Let $\mathcal{I} \subset \mathbb{R}$. Clearly, write out $f^{-1}(\mathcal{I})$, for the above f.

(e) Using your previous answer, what is $f^{-1}((1,8))$?

(f) **Using the definition** of an open set, prove or disprove: $f^{-1}(1,8)$ is open.

(3) Let $(a_n) : \mathbb{N} \to \mathbb{R}$ be the sequence whose terms are given by
$$a_n = \frac{3n-4}{n+2}.$$
Show that (a_n) converges.

(4) Prove or disprove: Let $(a_n) : \mathbb{N} \to \mathbb{R}$ be a convergent sequence then (a_n) is bounded.

(5) Given a set $Y \subset \mathbb{R}$ and function $f : \mathbb{R} \to \mathbb{R}$,

(a) Define the set $f^{-1}(Y)$

(b) Show that f is continuous at every $a \in \mathbb{R}$ if and only if $f^{-1}(A)$ is open for all open sets $A \subset \mathbb{R}$.

(6) Let $(z_n) : \mathbb{N} \to \mathbb{R}$ be a bounded sequence.

(a) Prove that the sequence defined by $g_n = \sup\{z_k : k \geq n\}$ is bounded.
(Note: You need to justify that such a sequence can even exist.)

(b) We define the <u>*greater limit*</u> of (z_n) by

$$g\lim z_n = \lim_{n\to\infty} g_n, \text{ where } g_n \text{ is defined above.}$$

Write a similar definition for the <u>*lesser limit*</u> $\ell\lim z_n$.

(c) Show: The sequences corresponding to the limits $\ell\lim z_n$ and $g\lim z_n$ are both monotone. (Hint: one of these sequences you defined and the other is (g_n).)

(d) Show: $\ell\lim z_n$ and $g\lim z_n$ exist.

(e) Prove: $\ell\lim z_n \leq g\lim z_n$.

(f) Show that $\ell\lim z_n = g\lim z_n$ if and only if $\lim\limits_{n\to\infty} z_n$ exists.

(g) Give an example where $\ell\lim z_n \neq g\lim z_n$.

(7) Let $S_n = \displaystyle\sum_{k=1}^{n} \frac{1}{3^k}$.

(a) Use the definition of convergence to show that the sequence $(S_n) : \mathbb{N} \to \mathbb{R}$ is convergent.

(b) Prove or disprove:

$$\text{The set } \bigcup_{n=1}^{\infty} [S_{2n-1}, S_{2n}] \text{ is compact.}$$