MATH 351 Fall 2015 Exam 1 Due: Tuesday 10/27

<u>**Directions:**</u> You may use your textbook but not any solutions manual. You may use **your own** notes and **your own** homework. You may not use the internet or any other electronic or printed resource including calculators, mathematical software, or other books. You may not use anyone elses notes. If you work on this exam in a public space you must keep your work private. Do not leave your work where it is visible. Erase boards if you work on them. You may not discuss this exam with or near any person except professor Heather Moon. Failure to follow the letter and spirit of these instructions will result in you failing the course. Your solutions should be **typeset with LaTeX**. Be sure to check that each of your proofs is clear, correct and complete. Your audience is not Heather, nor is it Yingyi. You must provide sufficient reasoning to completely convince any of your classmates. Any significant results that you use must be cited. Problems from homework may be cited, but you must be able to prove them. If the problem is taken directly from a homework or class assignment, you must do the entire problem. The exam is due at 8 PM on Tuesday, October 27. Problems 1 and 2 are worth 20 points each and problems 3 and 4 are worth 30 points each.

(1) Show that

$$\bigcup_{0<t<1}(0,t)=(0,1).$$

(2) Suppose $(a_n):\mathbb{N}\to\mathbb{R}$ is a sequence that satisfies:

For all $\varepsilon>0$ there is an $N\in\mathbb{N}$ so that $|a_{n+1}-a_n|<\varepsilon$ for all $n\geq N$.

Prove or disprove: The sequence (a_n) converges.

(3) Choose **either** a or b below (if you do both, I will record the problem with the **lowest** score):

(a) This summer, I spent a month in Madrid, studying metric space analysis. While in Madrid I thought up the Madrid metric, so named because every distance is measured from a central point, Puerta del Sol. In the middle of Madrid is Puerta Del Sol and in the middle of Puerta Del Sol is a spot on the ground that says (translated) zero kilometers. All distances are measured from this point. Anyway, the Madrid metric works this way:

The distance between any two distinct places on Earth, is the sum of their euclidean distances to Puerta Del Sol. The distance between a place and itself is defined to be zero.

(i) Let d be the Madrid Metric. Show that $(\mathbb{R}^2,d)$ is a metric space, where Puerta Del Sol is a fixed point $M=(M_1,M_2)\in\mathbb{R}^2$.

(ii) Draw a picture of the neighborhood $\mathcal{N}_2(x,y)\subset\mathbb{R}^2$, where (x,y) is a distance of $1+\varepsilon$ (for some small $\varepsilon>0$) kilometers from M.

(iii) Prove or disprove: the sequence $(a_n):\mathbb{N}\to\mathbb{R}^2$ defined by $a_n=(\frac{1}{n},\frac{1}{n})$ converges in $(\mathbb{R}^2,d)$.

(b) When considering the amount of noise in a 1-dimensional signal, one might notice that the sum of the absolute difference between neighboring data points is larger when noise is present than when noise is not present. We can define the distance between two signals, S and T, of length n as

$$d(S,T) = \sum_{k=2}^{n} |(s_k - s_{k-1}) - (t_k - t_{k-1})| + \sum_{k=1}^{n} |s_k - t_k|,$$

where $S = (s_1, s_2, \ldots, s_n)$ and $T = (t_1, t_2, \ldots, t_n)$.

(i) Show that $(\mathbb{R}^n, d)$ is a metric space.

(ii) Describe, in words, what each sum for d above is measuring.

(iii) Write the neighborhood $\mathcal{N}_2(0) \subset \mathbb{R}^n$, where 0 is the vector of zeros in $\mathbb{R}^n$ as a set.

(iv) Prove or disprove: Let $a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ and let

$$s_k = \left(a_1, a_2 - \frac{a_3 - a_1}{2k}, a_3 - \frac{a_4 - a_2}{2k}, \ldots, a_{n-1} - \frac{a_n - a_{n-2}}{2k}, a_n\right).$$

Then (s_k) converges in $(\mathbb{R}^n, d)$.

(4) Choose **either** a or b below (if you do both, I will record the problem with the **lowest** score):

(a) Suppose the sequence $(a_n) : \mathbb{N} \to \mathbb{R}$ converges and suppose $a_k \geq a_{k+1}$.

(i) Show that $A = \{a_n | n \in \mathbb{N}\}$ has an infimum.

(ii) Prove or disprove: $\inf A = \lim_{n\to\infty} a_n$ (here convergence is based on the standard metric in $\mathbb{R}$).

(b) Suppose the sequence $(a_n) : \mathbb{N} \to \mathbb{R}$ is defined by $a_n = (1 + \frac{1}{n})^n$.

(i) Determine whether $A = \{a_n | n \in \mathbb{N}\}$ has a supremum. Prove your result.

(ii) Prove or disprove: $\sup A = \lim_{n\to\infty} a_n$ (here convergence is based on the standard metric in $\mathbb{R}$).