Calculus II, Spring '04 Name:____________________

Big Opportunity

Here you go! It's your last chance to show me what you've learned this semester. As before, you have as much time as you need, but you may not use a calculator, cell phone, or spare brain. You may pray to the (non-corporeal) higher power of your choice, but you don't really need to because you're all SMART! And you can do this! For all problems, explain your work - if you use a test or theorem, say which one.

1. Taylor Series! (20 points)

Write out the first four terms of the Taylor series (i.e. $T_3(x)$) for $\sin x$ around the point $a = \frac{\pi}{4}$.

Note that $\frac{\pi}{4}$ is close to 1 (it's within 1/4). Suppose you used $T_3(1)$ as an approximation for $\sin(1)$. Give an estimate of the error of this approximation.

Write out the whole Taylor Series in summation notation. (Hint: The sequence $1, 1, -1, -1, 1, 1, -1 - 1 \ldots$ can be written as $(-1)^{\frac{n(n-1)}{2}}$

2. Integrals! (10 points)

Do the following integral in two different ways. (You should get the same answer, right?)

$$\int \frac{x}{\sqrt{4 - x^2}}\, dx$$

3. Derivatives! (10 points)

Find $g'(x)$ in two different ways. Justify each step of your calculations.

$$g(x) = \int_0^x t \sin(t)\, dt.$$

4. Sequences! (15 points) Find limits of the following sequences (if they exist). Explain your answers.

$a_n = \frac{2n}{\sqrt{n^2+2}}$

$b_n = (1 - \frac{\sqrt{\pi}}{n})^{-n}$

$c_n = \frac{\sin n}{n}$

5. Series! (15 points) Do the following series converge or diverge? Why?

a) $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{\pi^n}$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n}$

c) $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{\sqrt{n^3+2}}$

6. Exponential Growth! (10 points)

Trouble with Tribbles. As you recall from this classic Star Trek episode (old sexist Trek, not Next Generation or Voyager, or the new crappy one with Scott Bakula), a small furry creature called a tribble is taking over the Enterprise. It all started when Uhura brings 3 tribbles on board. An hour later there were 7.

If we assume they grow exponentially, how many tribbles are on board 12 hours after the first batch arrives?

How long until there are a million tribbles on board? (If you would like an approximation of your answer, see me or Kristine.)

Explain why a logistic model might be more realistic in this case.

7. Volumes! (10 points)

Take the region between the functions $x = 0$, $y = 1$, and $y = \frac{1}{\sqrt{x}}$ (shown on the right) and spin it around the y-axis.

Is the resulting volume infinite or finite? Explain.

8. (10 points) You studied endless material for his exam. Undoubtedly, some of it didn't appear in the problems above. Write a problem you think should have been on this exam. Answer it. You will be graded both on the problem and the solution.

Note: You may **not** use a problem from a previous test (from this semester or previous semesters). Your problem also shouldn't be similar to a problem already on this Opportunity.

Extra Credit: In the 1950's, mathematician Kenneth Arrow proved a surprising theorem about voting systems. The only system that satisfies some measure of consistency is a dictatorship, where one person has all the power. A 17th century philosopher wrote about such a system, and called the all-powerful leader a "benevolent dictator." Name the philosopher and his most famous work.