Name: ______________________________

FINAL OPPORTUNITY   FALL '09

Say "Calculus Rocks!" out loud! Now!

No calculators or cell phones are allowed — please zip them away in your bookbag. If you have any questions, please ask Dave. Explaining your reasoning will help you earn partial credit if your answer isn't entirely correct. Please write clearly and legibly; scratch paper will be available, but you should only turn in the exam.

Here are a couple of series that might prove useful:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

1. The definition of the natural log function is $\ln(x) = \int_1^x \frac{1}{t}\, dt$.

Using the Fundamental Theorem of Calculus, prove the following:

$(\ln x)' = \frac{1}{x}$

$(\ln x^2)' = \frac{2}{x}$ (Hint: use the Chain Rule)

Explain how to use the Fundamental Theorem of Calculus to calculuate the area under the curve $y = \sin x$ between $x = 0$ and $x = \pi$. Be sure to note exactly how you are using the FTC.

2. Here's a rough sketch of the function $f(x) = xe^{-x^2}$.

Is the area under f in the first quadrant finite or infinite? Support your conclusion with a detailed calculation.

3. Suppose we look at the region under this same function f between $x = 0$ and $x = 1$. Write an integral that calculates the volume formed when this region is rotated around the x-axis.

4. True or False (write the entire word "True" or the entire word "False" after each statement). If you write "False", give a counterexample.

a) If $\sum_{k=1}^{\infty} a_k$ converges then $a_k \to 0$.

b) If $a_k \to 0$ then $\sum_{k=1}^{\infty} a_k$ converges.

c) If $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \frac{1}{2}$ then $\sum_{n=1}^{\infty} a_n$ converges.

5. Draw the direction field (a.k.a. slope field) for the differential equation $\frac{dy}{dx} = x + y$:

On your slope field, draw three solutions, one for each of the following initial values: $y(0) = 1$, $y(0) = 0$, $y(0) = -2$.

6. The goal of this problem is to find an approximation for the area under the function $g(x) = e^{-x^2}$ between $x = 0$ and $x = 1$.

[I hear you say, "Dave, let's just find an anti-derivative for g and then use the FTC!" Unfortunately, g has *no* anti-derivative that you can write down easily. Sucks for us. Luckly, power series come to the rescue!]

Write at least the first 3 non-zero terms of the Maclaurin Series for g (i.e. the Taylor Series for g centered at $x = 0$.) Note: there are two ways to do this, the easy one and the hard one. Please choose the easy one.

Use this approximation of g to approximate the integral in question. (That is, instead of integrating g, integrate its Maclaurin Series.)

Here's the hardest question on the test: How far off might your approximation be? Explain your reasoning.

7. Solve the following differential equation with the stated initial conditions.

$$y' = y \cos x \quad y(0) = \pi$$

8. Let's return to the differential equation in problem 5. Use power series to approximate the solution to this differential equation with the stated initial conditions. Be sure to calculate at least the first five terms of the series (through the x^4 term).

$$y' = x + y \quad y(0) = 1$$

Extra Credit: Where is President Obama flying today? Why is he going there? What country will he be most closely negotiating with?