Mid-Term Exam

Instructions: You may use proven results from the *class* textbook, you may consult your notes, and you may use assumptions and facts about sets learned in F.O.M. and discussed in this class. No other resources are allowed. If a theorem or result in the textbook is needed for your solution and this result has not yet been proved, you may cite the result (but you will not receive full credit unless you include a proof for the cited item; material from the additional sections can be cited). This exam is due at the start of class on March 19th.

The so-called "quickie problems" do not require rigorous explanation. Pictures and logical sketches and ideas suffice. The "proofy problems" are to be proven rigorously, and don't forget to find and complete your individual problems!

Quickies

Quickie Problem 1.
The subset

$$S^1 \vee S^1 = \left\{(x,y) \in \mathbb{R}^2 : (x+1)^2 + y^2 = 1 \text{ or } (x-1)^2 + y^2 = 1\right\} \subset \mathbb{R}^2$$

is called *the wedge of two circles* when it is equipped with the subspace topology from $\mathbb{E}^2$. Sketch a picture of $S^1 \vee S^1$ and explain whether or not it is homeomorphic to S^1.

Quickie Problem 2.
Is the subspace $\mathbb{Q} \subset \mathbb{E}$ connected? What about the subspace $\mathbb{Q} \times \mathbb{Q} \subset \mathbb{E}^2$?

Quickie Problem 3.
Professor Brandtwells claims that

$$[0,1] \times [0,1] \approx [0,1) \times [0,1)$$

as subspaces in $\mathbb{E}^2$. Aside from the fact that he is Brandtwells, how do you know that he is wrong?

Quickie Problem 4.
Explain how every (filled in) triangle in the plane is homeomorphic to every closed disk, all with respect to the Euclidean topology. In particular, find an explanation that uses the the following fact: any two closed line segments are homeomorphic.

Proofy Problems

Problem 1.

(a) Prove that a function $f : X \to Y$ between two topological spaces is continuous $\iff$ $f^{-1}(C) \subseteq X$ is closed for every closed $C \subseteq Y$.

(b) Prove that a function $f : X \to Y$ between two topological spaces is continuous $\iff$

$$f\left(Cl(S)\right) \subseteq Cl(f(S))$$

for every set $S \subseteq X$.

Problem 2.
Suppose $f : X \to Y$ is a map between topological spaces. Is it true that if C is a connected subspace $C \subseteq Y$ then $f^{-1}(C)$ is connected in X? Prove your answer.

Problem 3.
This problem requires the following definition.

Definition A *subbasis* $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals X. The *topology generated by the subbasis* $\mathcal{S}$ is given by using the basis

$$\mathcal{B} = \{U \mid U = \text{ the intersection of finitely many elements from } \mathcal{S}\}.$$

(a) Prove that $\mathcal{B}$ is, in fact, a basis.

(b) Prove that the collection

$$\mathcal{S} = \{(-\infty, a), (b, \infty) \mid a, b \in \mathbb{R}\}$$

forms a subbasis for $\mathbb{E}$.

(c) What happens if we take as a subbasis

$$\mathcal{S} = \{(-\infty, a) \mid a \in \mathbb{R}\}?$$

Alex, Monica, and Hattie

Problem 4.
Hattie has been quoted as saying "Single point sets that are open are just weird." Given that and Monica and Alex's excellent work on Lemma 3.30, the following problem should be a dream.

(a) Let X be a Hausdorff space. Prove that all single-element subsets are closed.

(b) Suppose further that X is a compact, Hausdorff space. Suppose $A \subset X$ is closed and that $x \notin A$. Prove that there exist disjoint, open sets U and V such that $x \in U$ and $A \subset V$.

(c) Again suppose X is a compact, Hausdorff space and suppose $A, B \subset X$ are closed, disjoint sets. Prove that there exist disjoint, open sets U and V such that $A \subset U$ and $B \subset V$.

By the way, spaces with these various properties have special names. A space in which every pair of closed sets can be separated by open sets is called *normal*, while a space in which every point and every closed set can be separated is called *regular*.

Lexi

Problem 5.
Prove that every subset in $\mathbf{F^1}$ is compact.

Kayla and Ian

Problem 6. *The Pasting Lemma*
Let $X = A \cup B$ where A and B are closed in the topological space X, and let Y be another topological space. Let $f : A \to Y$ and $g : B \to Y$ be continuous. Prove that if $f(x) = g(x)$ for every $x \in A \cap B$, then the "pasted function"

$$h(x) = \begin{cases} f(x) \text{ if } x \in A \\ \\ g(x) \text{ if } x \in B \end{cases}$$

is continuous as a function $h : X \to Y$.

Dan, Jackson, and Lawrence

Problem 7. A Fixed-Point-ish Theorem
Let X be any topological space, and suppose Y is a Hausdorff space. Let $f, g : X \to Y$ be continuous functions. Prove that the set

$$S = \{x \in X \mid f(x) = g(x)\}$$

is closed. What does this result imply about the fixed point set of a map $f : X \to X$ (where, again, X is Hausdorff)?

Jeremy and Josh

Problem 8.
Define the set X to be the Cartesian product of a real-line's worth of real lines. This set can also be regarded as the set of all functions with domain and co-domain equal to $\mathbb{R}$:

$$X = \mathbb{R}^{\mathbb{R}} = \{f : \mathbb{R} \to \mathbb{R}\} = \prod_{\alpha \in \mathbb{R}} \mathbb{R}$$

(a) Prove that the collection of subsets

$$\mathcal{B} = \left\{ \prod_{\alpha \in \mathbb{R}} U_\alpha \,\middle|\, U_\alpha \text{ is open in } \mathbb{R} \text{ and } U_\alpha = \mathbb{R} \text{ for all but finitely many } \alpha \in \mathbb{R} \right\}.$$

can be used as a basis for a topology on $X = \prod_{\alpha \in \mathbb{R}} \mathbb{R}$ This topology is called the *product topology* on X. As an example, the set

$$U = \prod_{\alpha \in \mathbb{R}} U_\alpha$$

where $U_1 = (-1, 1), U_\pi = (-\infty, 2), U_{10e} = (0, 2) \cup (10, \infty), U_{2013} = \mathbb{R}\backslash\{0\}$, and $U_\alpha = \mathbb{R}$ for all $\alpha \in \mathbb{R}\backslash\{1, \pi, 10e, 2013\}$ is an open set; actually, it is a particular basis element.

Suggestion: Elements in and subsets of X can be difficult to notate. You may want to write $X = \prod_{\alpha \in \mathbb{R}} \mathbb{R}_\alpha$ so that you can distinguish each factor.

(b) Prove that a sequence of points $\{f_n\} \subset X$ converges to a point $f \in X \iff$ the elements f_n, when viewed as functions $f_n : \mathbb{R} \to \mathbb{R}$, converge pointwise to the function f.

(c) Is X Hausdorff? Is X connected? Given the length of this problem, you are welcome to sketch your ideas for answering this question (as opposed to providing rigorous proofs).

Pasquale

Problem 9.
Recall that a function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is said to be *continuous at* $x_0 \in D$ if $\forall \epsilon > 0 \, \exists \delta > 0$ such that

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon.$$

(a) Prove that the set D in the above definition must be open (in $\mathbb{E}$).

(b) Prove that if f is continuous at *every* $x_0 \in D \iff f$ is continuous in the topological sense (with respect to $\mathbb{E}$ and the subspace topology).

Christiana and Caroline

Problem 10.
Suppose X is a connected space and Y is a discrete space. Prove or disprove: every map $f : X \to Y$ is constant.

Joe and Emily

Problem 11.

Theorem. Let X be a Hausdorff space. If the sequence $\{x_n\}$ converges to x and y, then ...

Complete the above theorem and prove it.

Kevin

Problem 12.
Prove Theorem 4.8. You may use any of the preceding results in Chapter 4 without penalty.