6 Manifolds, Group Presentations

Here we look at a special class of spaces that share many properties of Euclidean spaces. We also introduce some useful algebraic notation.

Definition Let $\mathbb{E}^n_+ = \{(x_1, \cdots, x_n) \in \mathbb{E}^n | x_n \geq 0\}$.

Definition A Hausdorff space X is an *n-manifold* $\Leftrightarrow \forall x \in X, \exists U_x$ open in X containing x and a homeomorphism $f_x : \mathbb{E}^n \to U_x$ or $\mathbb{E}^n_+ \to U_x$ such that $f_x(\mathbf{0}) = x$. The set of points for which $f_x : \mathbb{E}^n \to U_x$ is called the *interior* of X and is denoted $\text{int}(X)$. The set of x for which $f_x : \mathbb{E}^n_+ \to U_x$ we call the *boundary* of X and denote it by $\text{bnd}(X)$. A 2-manifold is usually called a *surface.*

Theorem 6.1. *If X is a surface then int(X) $\cap$ bnd(X) $= \emptyset$.*

Proof.

Prove . . . let $x \in \text{int}(X) \cap \text{bnd}(X)$, there exist two open sets U_x and V_x and homeomorphisms f_x, g_x . . . think about something like $h_x = g_x^{-1} \circ f_x$ which doesn't quite make sense but .... think about what this implies about $E^2 \setminus \{0\}$ and $E^2_+ \setminus \{0\}$ . . . □

Theorem 6.2. *Let X and Y be surfaces such that $X \approx Y$. Then int(X) $\approx$ int (Y) and bnd(X) $\approx$ bnd(Y).*

Proof.

Prove. □

Example 6.3.

Does the theorem above help with classifying any of our spaces?

Definition Given a finite set X, we define $[X]$ as follows:

- The elements of X are called generators.
- A word in X is a finite product $x_1^{n_1} x_2^{n_2} \cdots x_n^{n_n}$ where each x_i is in X and each n_i is an integer.
- A word is reduced if x_i never equals x_{i+1} and all the n_i are non-zero.
- Given any word we can reduce it by collecting powers of adjacent elements and replacing the word x_i^0 with the empty word $\emptyset$.
- $[X] = \{$All reduced words in $X\}$.
- We multiply elements of $[X]$ by simply writing them next to each other and reducing the resulting word.

Theorem 6.4. $[X]$ *is a group.*

Proof.

Prove. □

Definition We call $[X]$ the *free group generated by* X.

Remark 6.5. $[x_1, x_2 \cdots x_n] \cong [y_1, y_2 \cdots y_n]$.

Definition A *group with relators* $[X|R]$ is defined in an analogous way to a free group:

- Start with a set of generators $X = \{x_1, x_2, \cdots x_n\}$.
- Let $R = \{r_1, r_2, \cdots r_m\}$ be a set of words in $[X]$. These are called *relators*.
- Elements of $[X|R]$ are again reduced words in X except that when a relator r_i appears in a word we can set $r_i = \emptyset$ or we can insert r_i at any point.
- Note that if $x \in X$ and $r = x^3 \in R$ we can replace x^3 by $\emptyset$ or x^2 by x^{-1}.
- The symbol $[x_1, x_2, \cdots x_n | r_1, r_2, \cdots r_m]$ is called a *presentation for the group*.
- Note that if R is empty then the group is just the free group $[X]$.

Example 6.6. $[x], [x|x^3], [x, y|x], [x, y],$ and $[x, y|xyx^{-1}y^{-1}]$ are all presentations for (somewhat) familiar groups.

Identify these groups.

Example 6.7.

Give presentations for the fundamental groups of each of the following spaces. In each case draw a picture of the space and on the picture draw the image of a representative loop from the equivalence class of each generator. $\mathbf{S}^1, \mathbf{S}^2,$ Mobius Band , $\mathbf{S}^1 \times [0,1], \mathbb{E}^2 \setminus \{0\},$ and $\mathbf{S}^1 \times \mathbf{S}^1$.

Definition If $G = [X|R]$ and $H = [Y|S]$ are two groups then the *free product of* G *and* H, denoted by $G \star H = [X \uplus Y | R \uplus S]$. In the case where X and Y contain some of the same symbols, we usually rename them first so that $X \uplus Y = X \cup Y$.

Theorem 6.8. *If* X *is a space with* $X = A \cup B$ *where* $A \cap B = \{x\}$ *then* $\Pi_1(X, x) \cong \Pi_1(A, x) \star \Pi_1(B, x)$.

Proof.

Prove. Actually an intuitive argument is OK here □

Example 6.9.

Give presentations for the fundamental groups of each of the following spaces. In each case draw a picture of the space and on the picture draw the image of a representative loop from the equivalence class of each generator. A figure eight, $\mathbf{B}^2 \setminus \{2 \text{ copies of } \mathbf{B}^2\}$, and $\mathbf{S}^1 \times \mathbf{S}^1 \setminus \{1 \text{ copy of } \mathbf{B}^2\}$.

Example 6.10. Recall that if $A \subset X$ and $i : A \to X$ is the inclusion map $i(a) = a, \forall a \in A$ then $\hat{i} : \Pi_1(A) \to \Pi_1(X)$ is a group homomorphism.

For each of the following pairs X, A

- Find a presentation of $\Pi_1(A)$ and $\Pi_1(X)$. Hint: If the two groups are the same, use different names for the generators.
- Draw a picture of the image under i of a loop from each generating class in $\Pi_1(A)$.
- Write the image under $\hat{i}$ of each generator of $\Pi_1(A)$ in terms of the generators of $\Pi_1(X)$.

X	A
$\mathbf{S}^2$	The equator
$\mathbb{E}^2 \setminus \{0\}$	$\mathbf{S}^1$
$\mathbf{B}^2$	The boundary circle
$\mathbf{S}^1 \times [0,1]$	One of the boundary circles
Mobius Band	The boundary circle
$(\mathbf{S}^1 \times \mathbf{S}^1) \setminus \mathbf{B}^2$	The boundary circle

Theorem 6.11. *(Van Kampen) Let $X = A \cup B$, A,B open in X, $A \cap B$ be path connected, and $i : A \cap B \to A$, and $j : A \cap B \to B$ be the inclusion maps. Then $\Pi_1(X) \cong \Pi_1(A) \star \Pi_1(B)$ with the additional relations that $\hat{i}(\langle\alpha\rangle) = \hat{j}(\langle\alpha\rangle)$ for all $\langle\alpha\rangle$ generators of $\Pi_1(A \cap B)$.*

Proof.

Prove. Give an intuitive argument for the proof of this. □

Example 6.12.

Use the Van Kampen Theorem to compute the fundamental group for each of the following spaces: Torus, Projective Plane, Klein Bottle, Double Torus.

Example 6.13.

Do we have any pairs of spaces left that we can't distinguish?