5 The Fundamental Group

In this chapter we show how to associate a group with a topological space. When the spaces are the same the groups will be the same.

Definition A *path* in a topological space X is a map $\alpha : [0,1] \to X$. The points $\alpha(0)$ and $\alpha(1)$ are said to be *joined* by α. A space is *path-connected* if any two points can be joined by a path.

Theorem 5.1. *$X \approx Y \Rightarrow$ (X is path-connected $\Leftrightarrow$ Y is path-connected).*

Proof.

Prove. □

Theorem 5.2. *Every path-connected space is connected.*

Proof.

Prove, but show that the converse is false. □

Definition Given a space X and a point x in X, a *loop based at x* is a path α with $\alpha(0) = \alpha(1) = x$.

Example 5.3.

In each of the following, choose an x in X and draw several loops in X based at x. Actually what you will draw is the image of the loop but we will use the same name for both. $\mathbb{E}^1$, $\mathbb{E}^2$, $\mathbf{S}^1$, $\mathbb{E}^2 \setminus \mathbf{B}^2$, and $\mathbf{S}^1 \times \mathbf{S}^1$.

Definition If α and β are paths in a path-connected space X such that $\alpha(1) = \beta(0)$, define

$$\alpha \star \beta(s) = \begin{cases} \alpha(2s) & 0 \leq s \leq \frac{1}{2} \\ \beta(2s-1) & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Definition Given a path-connected space X and a point x in X define $L(X,x)$ = { all loops in X based at x} .

Theorem 5.4. *$(L(X,x), \star)$ forms a group.*

Proof.

Disprove. □

Definition Let α, β be paths connecting x and y in a space X. We will say that *α is homotopic to β* (written $\alpha \sim \beta$) $\Leftrightarrow$ there exists a map $H : [0,1] \times [0,1] \to X$ such that $H(s,0) = \alpha(s)$, $H(s,1) = \beta(s)$, $H(0,t) = x$, and $H(1,t) = y$. Note that when α and β are loops based at x, $H(0,t) = H(1,t) = x$.

Example 5.5.

In the spaces in Example 5.3 find some pairs of loops that are homotopic and some that are not.

Theorem 5.6. *The relation $\sim$ is an equivalence relation on $L(X,x)$.*

Proof.

Prove this by showing:

1. $\alpha \sim \alpha$
2. $\alpha \sim \beta \Rightarrow \beta \sim \alpha$
3. $\alpha \sim \beta$ and $\beta \sim \gamma \Rightarrow \alpha \sim \gamma$

$\square$

Definition Let $\langle\alpha\rangle$ denote the homotopy equivalence class of α, that is the collection of all loops homotopic to α. Then let $\Pi_1(X,x) = \{\langle\alpha\rangle | \alpha \in L(X,x)\}$ and $\langle\alpha\rangle \otimes \langle\beta\rangle = \langle\alpha \star \beta\rangle$.

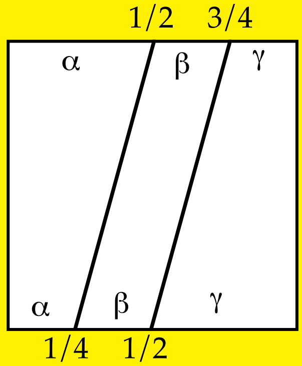
Theorem 5.7. $(\Pi_1(X,x), \otimes)$, *is a group.*

Proof.

Prove.

1. Show that the system is closed . . . this follows from the definition.

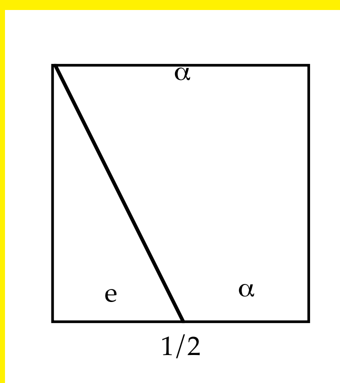
2. To show associativity, we need to show $(\langle\alpha\rangle \otimes \langle\beta\rangle) \otimes \langle\gamma\rangle = \langle\alpha\rangle \otimes \langle\beta\rangle \otimes \langle\gamma\rangle)$. To do this we need to show $(\alpha \star \beta) \star \gamma \sim \alpha \star (\beta \star \gamma)$ ...



here is a picture that may help....

□

3. $e : [0, 1] \to X$ by $e(t) = x$ (for all t) seems a likely candidate for the identity. Show $\langle e\rangle \otimes \langle\alpha\rangle = \langle\alpha\rangle \otimes \langle e\rangle = \langle\alpha\rangle$ . . . here is another



picture . . .

4. What should α^{-1} be? Try drawing the frames of a movie that show $\alpha \star \alpha^{-1} \sim e$ . . .

Definition We will call $\Pi_1(X, x)$ the *fundamental group of X based at x*.

Definition Recall the following definitions from algebra. If G and H are groups, a function $h : G \to H$ is a *homomorphism* $\Leftrightarrow \forall a, b \in G, h(a \otimes b) =$

$h(a) \otimes h(b)$. A bijective homomorphism is an *isomorphism*. Two groups are *isomorphic* (denoted $G \cong H$) $\Leftrightarrow$ there exists an isomorphism between them.

Definition A space X is *simply connected* $\Leftrightarrow$ for all $x \in X, \Pi_1(X, x) \cong \{\langle e \rangle\}$.

Theorem 5.8. *Any topological space with the trivial topology is simply connected.*

Proof.

Prove. To do this, show that if α is any loop in X, then $\alpha \sim e$. □

Definition Let $\mathbf{x}$ and $\mathbf{y}$ be two points in $\mathbb{E}^n$. We define the line segment $L(\mathbf{x}, \mathbf{y})$ between $\mathbf{x}$ and $\mathbf{y}$ as $L(\mathbf{x}, \mathbf{y}) = \{(1-t)\mathbf{x} + t\mathbf{y} | 0 \leq t \leq 1\}$. A subset A of $\mathbb{E}^n$ is said to be *convex* $\Leftrightarrow \forall \mathbf{x}, \mathbf{y} \in A, L(\mathbf{x}, \mathbf{y}) \subset A$.

Theorem 5.9. *Any convex subset of $\mathbb{E}^n$ with the subspace topology is simply connected.*

Proof.

Prove. □

Theorem 5.10. $\Pi_1(\mathbf{S}^1, (1, 0)) \cong \mathbb{Z}$.

Proof.

Give some intuitive argument as to why the theorem is true or false. □

Theorem 5.11. *If X is path connected and $x, y \in X$ then $\Pi_1(X, x) \cong \Pi_1(X, y)$.*

Proof.

Prove...let γ be a path connecting x and y ...consider $\widehat{\gamma} : \Pi_1(X, x) \to \Pi_1(X, y)$ defined by $\widehat{\gamma}(\langle \alpha \rangle) = \langle \gamma^{-1} \star \alpha \star \gamma \rangle$ ... $\widehat{\gamma}$ is an isomorphism ... □

Remark 5.12. Given the above theorem we will refer to $\Pi_1(X, x)$ as simply $\Pi_1(X)$ when X is path-connected.

Definition

Make up a definition for the term *star-shaped* so that the following theorem is true.

Theorem 5.13. *Any star-shaped subset of $\mathbb{E}^n$ with the subspace topology is simply connected.*

Proof.

Prove using the same proof as in Theorem 5.9 . □

Definition Given a map $f : X \to Y, x \in X, y \in Y$, and $f(x) = y$, we define a function $\widehat{f} : \Pi_1(X, x) \to \Pi_1(Y, y)$ by $\widehat{f}(\langle \alpha \rangle) = \langle f \circ \alpha \rangle$. We say that $\widehat{f}$ is *induced* by f.

Theorem 5.14. *$\widehat{f}$ as defined above is a group homomorphism.*

Proof.

Prove. ☐

Theorem 5.15. *Given maps $f : X \to Y$ and $g : Y \to Z$, $\widehat{(g \circ f)} = \widehat{g} \circ \widehat{f}$.*

Proof.

Prove. ☐

Theorem 5.16. $(X \approx Y) \Rightarrow (\Pi_1(X) \cong \Pi_1(Y))$.

Proof.

Prove . . . then by the previous theorem . . . ☐

Theorem 5.17. $(\Pi_1(X) \cong \Pi_1(Y)) \Rightarrow (X \approx Y)$.

Proof.

Prove or disprove. ☐

Theorem 5.18. *Let X be a space such that $X = A \cup B$ where A and B are open. Then any path connecting a point in $A \setminus B$ and a point in $B \setminus A$ must pass through $A \cap B$.*

Proof. This follows from Lebesgue's Lemma which we will not prove here. ☐

Theorem 5.19. *Let X be a path connected space such that $X = A \cup B$, where A and B are open and simply connected and $A \cap B$ is path-connected and non-empty, then X is simply connected.*

Proof.

Prove....choose a base point x in $A \cap B$, let α be a loop in X based at x . . . find a sequence of points $0 = t_0 < t_1 < t_2 < t_3 < \cdots < t_n = 1$ such that $\alpha([t_i, t_{i+1}])$ is contained in A or B . . . $\alpha \sim e$ . . . ☐

Theorem 5.20. $\mathbf{S}^n$ *is simply connected for* $n \geq 2$.

Proof.

Prove. ☐

Theorem 5.21. $\mathbb{E}^3 \setminus \{0\}$ *is simply connected.*

Proof.

Prove. ☐

Theorem 5.22. $\mathbb{E}^2 \setminus \{0\}$ *is simply connected.*

Proof.

Prove or disprove. Does the same proof work here as in the previous theorem? ☐

Theorem 5.23. *If X and Y are path-connected spaces,* $\Pi_1(X \times Y) \cong \Pi_1(X) \times \Pi_1(Y)$.

Proof.
Prove ... define $\Psi : \Pi_1(X \times Y) \to \Pi_1(X) \times \Pi_1(Y)$ by $\Psi(\langle \alpha \rangle) = (\langle p_1 \circ \alpha \rangle, \langle p_2 \circ \alpha \rangle)$ ... □

Example 5.24.
With which spaces does the previous theorem help us?

Definition A subset A of a space X is a *retract* of $X \Leftrightarrow$ there exists a surjective map $g : X \to A$ such that $\forall a \in A, g(a) = a$ and a map $G : X \times [0,1] \to X$ such that:

$$\begin{array}{ll} G(x,0) = x & \forall x \in X \\ G(x,1) = g(x) & \forall x \in X \\ G(a,t) = g(a) = a & \forall a \in A, \forall t \in [0,1] \end{array}$$

The map g is called a *retraction map* and G is called a *homotopy.*

Example 5.25.
$\mathbf{S}^n$ is a retract of $\mathbf{B}^{n+1} \setminus \{0\}$.
$\mathbf{S}^n$ is a retract of $\mathbb{E}^{n+1} \setminus \{0\}$.
Explicitly write down the retraction map and the homotopy in both cases.

Example 5.26.
$\mathbf{S}^1$ is a retract of $\mathbf{S}^1 \times [0,1]$.
$\mathbf{S}^1$ is a retract of the Mobius Band.
A figure eight is a retract of $\mathbf{B}^2 \setminus \{$ two copies of $\mathbf{B}^2\}$.
In these cases just show several frames of the movie as X "retracts" onto A.

Example 5.27.
A figure eight is a retract of $\mathbf{S}^1 \times \mathbf{S}^1 \setminus \{\mathbf{B}^2\}$.
A figure eight is a retract of a Klein Bottle $\setminus \{\mathbf{B}^2\}$.
$\mathbf{S}^1$ is a retract of $\mathbf{P}^2 \setminus \{\mathbf{B}^2\}$.
Do the same as above. Hint: Start with a square with sides identified.

Theorem 5.28. *If A is a retract of X then* $\Pi_1(X) \cong \Pi_1(A)$.

Proof.
Prove ... think about $\widehat{g}$. □

Example 5.29.
List the fundamental groups that this theorem helps us find. Does it help us distinguish any of our remaining spaces?