4 Product spaces, Identification spaces

Here we will look at two ways to construct new spaces from existing spaces. We then look at which properties of the old spaces are inherited by the new ones.

Definition Let X and Y be topological spaces and define $\Delta = \{U \times V \subseteq X \times Y | U \text{ is open in } X \text{ and } V \text{ is open in } Y\}$.

Theorem 4.1. *The collection Δ satisfies the hypotheses of Theorem 1.9 and is thus a basis for a topology on $X \times Y$.*

Proof.

Prove. □

Definition We call this topology, $\mathcal{T}_{XY}$, the *product topology* and the resulting space the *product space*. The elements of Δ are called basic open sets.

Example 4.2.

Draw pictures of: $\mathbb{E}^1 \times \mathbb{E}^1$, $\mathbb{E}^1 \times [0,1]$, $[0,1] \times [0,1]$, $\mathbb{E}^1 \times \mathbf{S}^1$, $\mathbf{S}^1 \times \mathbf{S}^1$ (Note that the circles don't necessarily have radius 1) On each picture draw some basic open sets and some more general open sets.

Definition The functions $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ defined by $p_1(x,y) = x$ and $p_2(x,y) = y$ are called *projections*.

Theorem 4.3. *Projections are continuous and surjective.*

Proof.

Prove or disprove. □

Theorem 4.4. *Projections take open sets to open sets.*

Proof.

Prove or disprove. □

Theorem 4.5. *Projections take closed sets to closed sets.*

Proof.

Prove or disprove. □

Theorem 4.6. *A function $f : Z \to X \times Y$ is continuous $\Leftrightarrow$ the composite functions $p_1 \circ f : Z \to X$ and $p_2 \circ f : Z \to Y$ are continuous.*

Proof.

Prove. □

Theorem 4.7. *The function $D : X \to X \times X$ defined by $D(x) = (x,x)$ is a map.*

Proof.
Prove. ∎

Definition The function D is called the *diagonal map.*

Theorem 4.8. *X is Hausdorff $\Leftrightarrow$ $D(X)$ is closed in $X \times X$.*

Proof.
Prove. ∎

Theorem 4.9. *The product space $X \times Y$ is connected $\Leftrightarrow$ X and Y are connected.*

Proof.
Prove ... let $Z_{x,y} = (x \times Y) \cup (X \times y)$ ... by Thm 3.15 ... by Thm 3.15 again ... ∎

Example 4.10. Consider the following subsets of $\mathbb{E}^1 \times \mathbb{E}^1$.
Which are connected and why?
The set of all points in the plane with:

1. two rational coordinates.
2. at least one rational coordinate.
3. exactly one rational coordinate.
4. no rational coordinates.

Lemma 4.11. *(The Tube Lemma) Consider the product space $X \times Y$ with Y a compact space. If N is an open set in $X \times Y$ that contains the set $x_0 \times Y$, then N contains some set $W \times Y$ where W is an open set in X containing x_0.*

Definition A set of the form $W \times Y$ where $x_0 \in W$ is called *a tube* about $x_0 \times Y$.

Proof. (of the Tube Lemma)

First we observe that Y is homeomorphic to the space $x_0 \times Y$, where the latter is endowed with the subspace topology and the homeomorphism is given by $f(y) = (x_0, y)$. Since Y is assumed to be compact, we learn that $x_0 \times Y$ is a compact subspace of $X \times Y$.

Since N is an open set in $X \times Y$ it is, by definition, a union of basis elements for the product topology on $X \times Y$. These basis elements cover N and so they can also be used to cover the subspace $x_0 \times Y$, but the compactness of this space implies that some finite sub-collection of basis elements can be used as a cover: □

$$x_0 \times Y = (U_1 \times V_1) \cup (U_2 \times V_2) \cup \cdots \cup (U_n \times V_n)$$

Moreover, we can assume each set $U_i \times V_i$ intersects $x_0 \times Y$ (since, otherwise, it may be discarded from the open cover).

Define $W = U_1 \cap U_2 \cap \cdots \cap U_n$. Why is W open and why is $x_0 \in W$? In addition to answering these questions, prove that the sets $U_i \times V_i$ actually cover the tube $W \times Y$. Explain why $W \times Y \subset N$.

Theorem 4.12. *If X and Y are compact spaces, then $X \times Y$ is compact.*

Proof.

Given an open cover $\mathcal{A}$ of $X \times Y$, use a finite subcollection, $\{A_1, \ldots, A_m\}$, to cover the set $x_0 \times Y$, where $x_0 \in X$ is arbitrary. Apply the Tube Lemma to the set $N = A_1 \cup \cdots \cup A_m$ to conclude the existence of an open set W_{x_0}. Repeat this process to cover X by open sets W_x for each $x \in X$. The compactness of X ultimately implies that $X \times Y$ can be covered by finitely many tubes $W_i \times Y$. How many open sets of $\mathcal{A}$ are needed to cover each tube? □

Definition Let X be a space. A *partition* of X is a collection $\mathcal{P} = \{P_\alpha\}$ of disjoint non-empty subsets of X such that $\cup P_\alpha = X$.

Definition Given a space X and a partition $\mathcal{P}$ of X we form the *identification space* Y as follows. The points of Y are the elements of $\mathcal{P}$. To define the open sets in Y, let $\pi : X \to Y$ be the function that takes each point x of X to the element of $\mathcal{P}$ that contains x. We then say that O is open in $Y \Leftrightarrow \pi^{-1}(O)$ is open in X. This is called the *identification topology* for Y, and π is called the *identification map*.

Example 4.13. Let $X = \{a, b, c\}, T = \{\emptyset, \{a\}, \{a, b\}, X\}, \mathcal{P} = \{\{a, b\}, \{c\}\}$ and Y the identification space.

List the open sets of Y.

Example 4.14. Let $X = \mathbb{R}^1, \mathcal{T} = \mathbb{E}^1, \mathcal{P} = \{\{1\}, \{2\}, \mathbb{R}^1 \setminus \{1, 2\}\}$and Y the identification space.

List the open sets of Y.

Theorem 4.15. *Suppose X is a Hausdorff space and $\mathcal{P}$ is any partition, then the resulting identification space is Hausdorff.*

Proof.

Prove or disprove. □

Theorem 4.16. *Suppose X is a connected space and $\mathcal{P}$ is any partition, then the resulting identification space is connected.*

Proof.

Prove or disprove. □

Example 4.17. Let $X = [0, 1] \times [0, 1]$.

For each of the following, draw a picture of Y including a representative open set.

1. Let $\mathcal{P}$ consist of two types of subsets:
   (a) Sets consisting of a single point $\{(x, y)\}$ where $0 < x < 1$, $0 \leq y \leq 1$.
   (b) Sets consisting of pairs of points $\{(0, y), (1, y)\}$ where $0 \leq y \leq 1$.
2. Let $\mathcal{P}$ consist of two types of subsets:
   (a) Sets consisting of a single point $\{(x, y)\}$ where $0 < x < 1$, $0 \leq y \leq 1$.
   (b) Sets consisting of pairs of points $\{(0, y), (1, 1 - y)\}$ where $0 \leq y \leq 1$.
3. Let $\mathcal{P}$ consist of four types of subsets:
   (a) Sets consisting of a single point $\{(x, y)\}$ where $0 < x < 1$, $0 < y < 1$.
   (b) Sets consisting of pairs of points $\{(0, y), (1, y)\}$ where $0 < y < 1$.
   (c) Sets consisting of pairs of points $\{(x, 0), (x, 1)\}$ where $0 < x < 1$.
   (d) The set consisting of the four points $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$.
4. Let $\mathcal{P}$ consist of four types of subsets:
   (a) Sets consisting of a single point $\{(x, y)\}$ where $0 < x < 1$, $0 < y < 1$.
   (b) Sets consisting of pairs of points $\{(0, y), (1, 1 - y)\}$ where $0 < y < 1$.
   (c) Sets consisting of pairs of points $\{(x, 0), (x, 1)\}$ where $0 < x < 1$.
   (d) The set consisting of the four points $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$.
5. Let $\mathcal{P}$ consist of three types of subsets:
   (a) Sets consisting of a single point $\{(x, y)\}$ where $0 < x < 1$, $0 < y < 1$.

(b) Sets consisting of pairs of points $\{(0, y), (1, 1 - y)\}$ where $0 \leq y \leq 1$.

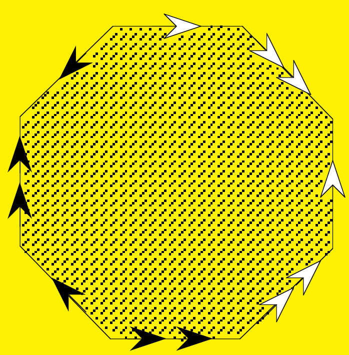
(c) Sets consisting of pairs of points $\{(x, 0), (1 - x, 1)\}$ where $0 < x < 1$.

Example 4.18. Let $X = \mathbf{B}^n$.

For each of the following, draw a picture of Y for $n = 1, 2$. What do you think happens when $n = 3, 4 \ldots$?

1. Let $\mathcal{P}$ consist of two types of subsets:
   (a) Sets consisting of a single point $\{\vec{x}\}$ where $|\vec{x}| < 1$.
   (b) The single set consisting of all $\{\vec{x}\}$ where $|\vec{x}| = 1$.
2. Let $\mathcal{P}$ consist of two types of subsets:
   (a) Sets consisting of a single point $\{\vec{x}\}$ where $|\vec{x}| < 1$.
   (b) Sets consisting of pairs of points $\{\vec{x}, -\vec{x}\}$ where $|\vec{x}| = 1$.

Example 4.19.



What is it?

Definition Given two sets A and B, define the *disjoint union* of A and B, written as $A \uplus B$ to be the regular union of A and B except that if a point is in both A and B it will appear twice in $A \uplus B$. e.g. $\{a, b, c\} \uplus \{b, c, d\} = \{a, b, c, b, c, d\}$.

Definition Given two spaces X and Y, a subset A of X and a map $f : A \to Y$, form a partition $\mathcal{P}$ of $X \uplus Y$ consisting of three types of subsets:

1. Sets consisting of a single point $\{x\}$ where $x \in X \setminus A$.
2. Sets consisting of a single point $\{y\}$ where $y \in Y \setminus f(A)$.

3. Sets consisting of pairs of points $\{a, f(a)\}$ where $a \in A$.

Let Z be the corresponding identification space and denote Z by $X \cup_f Y$. The map f is called an *attaching map*.

Example 4.20.

For each of the following, describe or draw a picture of $X \cup_f Y$.

1. $X = Y = [0, 1]$, $A = \{0, 1\}$, $f(0) = 0$, and $f(1) = 1$.
2. $X = Y = [0, 1]$, $A = \{0, \frac{1}{2}, 1\}$, and $f(a) = a$.
3. $X = Y = \mathbf{S}^1 \times [0, 1]$, $A = \{(s, 0)\} \cup \{(s, 1)\}$, and $f(a) = a$.

Example 4.21.

For each of the following, describe or draw a picture of $X \cup_f Y$.

1. $X = Y = \mathbf{B}^{n+1}$, $A = \mathbf{S}^n$ (the boundary of X), and $f(a) = a$, $n = 0, 1, 2 \ldots$???.
2. $X = Y = \mathbf{B}^2 \times \mathbf{S}^1$, $A = \{(x, y) \in \mathbf{B}^2 \times \mathbf{S}^1 | |x| = 1\}$ (Note $A = \mathbf{S}^1 \times \mathbf{S}^1$), and $f : A \to Y$ by $f(x, y) = (x, y)$.
3. $X = Y = \mathbf{B}^2 \times \mathbf{S}^1$, $A = \{(x, y) \in \mathbf{B}^2 \times \mathbf{S}^1 | |x| = 1\}$ (Note $A = \mathbf{S}^1 \times \mathbf{S}^1$), and $f : A \to Y$ by $f(x, y) = (y, x)$. (Note change in order.)
4. X = Möbius Band, $Y = \mathbf{B}^2$, $A = \mathbf{S}^1$ "boundary" of X, and $f : A \to \mathbf{S}^1$ "boundary" of Y be any homeomorphism.

By this point in the term you have undoubtedly been asked "What's Topology?" Give a one paragraph cocktail party answer.