3 Hausdorff and Connected Spaces

In this chapter we address the question of when two spaces are homeomorphic. This is done by examining two properties that are shared by any pair of homeomorphic spaces. More importantly, if two spaces do not share the property they are not homeomorphic.

Definition A space X is *Hausdorff* $\Leftrightarrow \forall x, y \in X$ such that $x \neq y, \exists U, V$ open in X such that $x \in U, y \in V$ and $U \cap V = \emptyset$.

Example 3.1. Let $X = \{a, b, c\}$

Take the list of topologies from Example 1.6 and decide which if any are Hausdorff. Justify your answers.

Theorem 3.2.

State and prove a theorem about finite Hausdorff spaces. Note that your proof should use the fact that your space is finite.

Theorem 3.3. $\mathbb{E}$ *is Hausdorff.*

Proof.

Prove or disprove. □

Theorem 3.4. $[0, 1]$ *with the subspace from* $\mathbb{E}$ *is Hausdorff.*

Proof.

Prove or disprove. □

Theorem 3.5. $\mathbf{H^1}$ *is Hausdorff.*

Proof.

Prove or disprove. □

Theorem 3.6. $\mathbf{F^1}$ *is Hausdorff.*

Proof.

Prove or disprove. □

Theorem 3.7. $\mathbf{D^1}$ *is Hausdorff.*

Proof.

Prove or disprove. □

Theorem 3.8. $\mathbf{T^1}$ *is Hausdorff.*

Proof.

Prove or disprove. □

Theorem 3.9. *Any subspace of a Hausdorff space is Hausdorff.*

Proof.

Prove or disprove. ☐

Theorem 3.10. $X \approx Y \Rightarrow (X \text{ Hausdorff } \Leftrightarrow Y \text{ Hausdorff})$.

Proof.

Fill in proof. ☐

Remark 3.11.

- State the converse of this theorem. Prove or disprove it.
- State the contrapositive. Prove or disprove it.
- Does this theorem help in classifying $\mathbb{E}, \mathbf{H^1}, \mathbf{F^1}, \mathbf{D^1},$ and $\mathbf{T^1}$?

Definition A space X is *connected* $\Leftrightarrow$ X cannot be written as the union of two non-empty disjoint open sets.

Example 3.12. Let $X = \{a, b, c\}$

Take the list of topologies from Example 1.6 and decide which if any are connected. Justify your answers.

Theorem 3.13. *X is connected $\Leftrightarrow$ The only subsets of X that are open and closed are X and $\emptyset$.*

Proof.

Fill in proof. ☐

Theorem 3.14. *$[a, b]$ with the subspace topology from $\mathbb{E}$ is connected.*

Proof.

Fill in proof. You may use the fact that any non-empty subset of $\mathbb{R}$ that has an upper bound has a least upper bound. Start by assuming that $[a, b] = A \cup B$ where A and B are non-empty, disjoint and open and W.L.O.G. that $b \in B$. Note that A must have a least upper bound x. Now think about where x might be. ☐

Theorem 3.15. *If there exists a collection Ψ of subsets of X whose union is X with the additional properties that each member of Ψ is connected (as a subspace of X) and no two members of Ψ are disjoint, then X is connected.*

Proof.

Fill in proof ... let A be a non-empty subset of X that is open and closed ... for each member C of Ψ consider $C \cap A$ as a subset of the connected space C ... $A = X$ . . . ☐

Theorem 3.16. *$(0, 1], [0, 1), (0, 1)$ and $\mathbb{E}$ are connected.*

Proof.

Fill in proof. ☐

Theorem 3.17. *Suppose $f : X \to Y$ is a surjective map. If X is connected then Y is connected.*

Proof.

Fill in proof. ☐

Theorem 3.18. $\mathbf{S}^1$ *is connected.*

Proof.

Prove or disprove. ☐

Theorem 3.19. *Let $X = \{x \in \mathbb{R} | x \text{ is rational}\}$. Give X the subspace topology from $\mathbb{E}$, then X is connected.*

Proof.

Prove or disprove. ☐

Theorem 3.20. $\mathbf{H^1}$ *is connected.*

Proof.

Prove or disprove. ☐

Theorem 3.21. $\mathbf{F^1}$ *is connected.*

Proof.

Prove or disprove. ☐

Example 3.22.

Find a topology for $\mathbb{R}^1$ that is not connected and not Hausdorff.

Theorem 3.23. *Any subspace of a connected space is connected.*

Proof.

Prove or disprove. ☐

Corollary 3.24. $X \approx Y \Rightarrow (X \text{ connected } \Leftrightarrow Y \text{ connected})$.

Proof.

Fill in proof. ☐

Remark 3.25.

- State the converse of this theorem. Prove or disprove it.
- State the contrapositive. Prove or disprove it.
- Does this theorem help in classifying $\mathbb{E}, \mathbf{H^1}, \mathbf{F^1}, \mathbf{D^1}$, and $\mathbf{T^1}$?

Theorem 3.26. *If $f : X \to Y$ is a homeomorphism and $A \subset X$ then $(X \setminus A) \approx (Y \setminus f(A))$ with the subspace topologies from X and Y.*

Proof.

Prove or disprove. ☐

Remark 3.27.

Does this theorem help in classifying $S^1, (0,1), [0,1),$ and $[0,1]$?

3.1 Compactness

Definition A collection $\mathcal{A}$ of subsets of a space X is said to *cover* X, or to be a *covering* of X, if the union of elements of $\mathcal{A}$ equals X. It is called an *open covering* of X if its elements are open subsets of X.

Definition A space X is said to be *compact* if ...

complete the definition.

Definition A subset Y of a space X is said to be *compact* if ...

complete the definition.

Theorem 3.28.
Let Y be a subspace of X. Then Y is compact if and only if every covering of Y by sets open in X contains a finite subcollection covering Y.

Proof.

Complete the proof. ☐

Theorem 3.29.

State the theorem that is proved below.

Proof. Let Y be a closed subspace of the compact space X. Given a covering $\mathcal{A}$ of Y by sets that are open in X, we form an open covering $\mathcal{B}$ of X by adjoining to $\mathcal{A}$ the single open set $X\backslash Y$. That is

$$\mathcal{B} = \mathcal{A} \cup \{X\backslash Y\}.$$

Since X is compact, some finite subcollection of $\mathcal{B}$ covers X. If this subcollection contains the set $X\backslash Y$, then discard $X\backslash Y$; otherwise leave the subcollection alone. In either case, the resulting subcollection is a finite subcollection of $\mathcal{A}$ that covers Y. ☐

Lemma 3.30. *If Y is a compact subspace of the Hausdorff space X and $x_0 \notin Y$, then there exist disjoint open sets U and V of X such that $x_0 \in U$ and $Y \subseteq V$.*

Proof.

Prove or disprove ☐

Theorem 3.31. *Every compact subspace of a Hausdorff space is closed.*

Proof.

Prove or disprove ☐

Theorem 3.32. *If X is a compact space and $f : X \to Y$ is a map, then $f(X)$ is compact.*

Proof.

Prove

□

Corollary 3.33. *The Extreme Value Theorem*

Proof.

Deduce the E.V.T. as a consequence of Theorem 3.32

□

Definition A collection $\mathcal{C}$ of subsets of X is said to have the *finite intersection property* if for every finite subcollection

$$\{C_1, C_2, \dots, C_n\}$$

of $\mathcal{C}$, the intersection $C_1 \cap C_2 \cap \cdots \cap C_n$ is non-empty.

Theorem 3.34. *A topological space X is compact $\iff$ for every collection $\mathcal{C}$ of closed sets in X having the finite intersection property, the intersection*

$$\bigcap_{C \in \mathcal{C}} C \neq \emptyset.$$

Proof. Complete the proof below:

To prove X is compact, let $\mathcal{A}$ be an open covering of X. Then consider the collection of closed sets

$$\mathcal{C} = \{X \backslash A | A \in \mathcal{A}\}.$$

Why does it follow that

$$\bigcap_{C \in \mathcal{C}} C = \emptyset$$

and why does this imply that some finite subcollection of sets $C \in \mathcal{C}$ do *not* have the finite intersection property? What does *this* imply about the open sets A?

Conversely, suppose X is compact and let $\mathcal{C}$ be a collection of closed sets having the finite intersection property. How does this imply that the set of open sets

$$\mathcal{A} = \{X \backslash C \,|\, C \in \mathcal{C}\}$$

is *not* an open cover of X? What does this imply about the intersection of all closed sets C?

□

Example 3.35. $X = \mathbb{E}^1$

Provide an open cover of X that admits no finite subcover. Are the subsets $(a, b), (a, b], [a, b)$ compact? Prove that $[a, b]$ is compact.

Example 3.36. $X = \mathbb{E}^n, Y = N(0,1) \subset X$

Provide an open cover of Y that admits no finite subcover.

Theorem 3.37. *A subset $A \subseteq \mathbb{R}$ is compact in the finite complement topology $\iff$ ...*

Complete and prove the statement of this theorem. Formulate and prove a corresponding theorem for $\mathbf{D^1}$.

Theorem 3.38. *If $K \subset \mathbb{E}^n$ is compact, then K is closed and bounded.*

Proof.

Prove or disprove. Assuming that n-dimensional boxes $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n b_n] \subset \mathbb{E}^n$ are compact, prove the converse to the theorem above. We will prove that such boxes are compact in the next chapter. □

Example 3.39.

Use Theorem 3.38 and its converse to prove S^n is compact for all $n \in \mathbb{N}$.

Definition If X is a space, a point $x \in X$ is said to be an *isolated point* if the set $\{x\}$ is open.

Theorem 3.40. *Let X be a non-empty, compact, Hausdorff space. If X has no isolated points, then X is uncountable.*

Proof.

Prove or disprove. □

Theorem 3.41. *If $f : X \to Y$ is a map and $C \subseteq Y$ is compact, then $f^{-1}(C)$ is compact.*

Proof.

Prove or disprove. □

Theorem 3.42. *If $f : X \to Y$ is a bijective map with X compact and Y Hausdorff, then f is a homeomorphism.*

Proof.

Prove or disprove. □

Theorem 3.43. *Suppose X is a compact space and $A \subseteq X$ is an infinite subset. Then A has a limit point in X.*

Proof.

Complete the following proof: Suppose A has no limit point in X. Then, for each $x \in X$ there exists an open set U_x containing x but satisfying $(U_x \backslash \{x\}) \cap A = \emptyset$. The collection $\mathcal{A} = \{U_x : x \in X\}$ is an open covering for X. Why is it impossible for a finite subcollection of $\mathcal{A}$ to cover X? □