2 Maps and Homeomorphisms

As with most mathematical constructions, one of the first things to do with topological spaces is to consider functions between them that preserve the topological structure. This then leads to the idea of two spaces being topologically the same. We say that such spaces are homeomorphic.

Definition
Let A and B be sets and define $A \times B = \{(a, b)|a \in A, b \in B\}$. We then define a *function* f from A to B (written $f : A \to B$) as any subset of $A \times B$ in which every element of A appears exactly once.

Definition

- Define *injective(one-to-one)*, *surjective(onto)*, and *bijective*.
- Define what it means for a function $f : A \to B$ to be *invertible*.
- Define the *inverse* f^{-1} of an invertible function f.

Example 2.1.

Let $A = \{a, b, c\}$ and $B = \{d, e\}$. Give examples of functions $f : A \to B$ that are:

- surjective but not injective.
- injective but not surjective.
- neither.
- both.
- invertible.

Theorem 2.2. *Let A and B be finite sets. If there exists a bijection $f : A \to B$, then . . . complete the theorem with a statement about the sizes of A and B.*

Proof.

- Prove the theorem
- State the converse. Is it true? □
- State the contrapositive. Is it true?

Definition Given a function $f : A \to B$ and $C \subseteq B$ define $f^{-1}(C) = \{a \in A|f(a) \in C\}$. Note that use of this notation does not imply that f is invertible.

Definition Suppose X and Y are topological spaces. A function $f : X \to Y$ is *continuous* $\Leftrightarrow \forall O$ open in $Y, f^{-1}(O)$ is open in X. We will henceforth call a continuous function between topological spaces a *map*.

Example 2.3. Let $X = \{a, b, c\}$, $\mathcal{T}_X = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$ and $Y = \{d, e\}$, $\mathcal{T}_Y = \{\emptyset, \{d\}, \{d, e\}\}$.

Give examples of functions from X to Y that are:

- continuous and surjective.
- continuous and not surjective.
- not continuous.

Theorem 2.4. *If X and Y are spaces, $\mathcal{B}$ a basis for the topology on Y and $f : X \to Y$ a function, then f is a map $\Leftrightarrow f^{-1}(B)$ is open in X for all B in $\mathcal{B}$.*

Proof.

Fill in proof. □

Example 2.5. Let $\{X, \mathcal{T}_X\} = \{Y, \mathcal{T}_Y\} = \{\mathbb{R}, \mathbb{E}\}$.

Give examples of functions from X to Y that are:

- continuous and surjective and injective.
- continuous and surjective and not injective.
- continuous and not surjective and not injective.
- continuous and injective and not surjective.
- not continuous.

Justify these using our definition.

Theorem 2.6. *If X,Y and Z are spaces and $f : X \to Y$ and $g : Y \to Z$ are maps, then $g \circ f : X \to Z$ is a map.*

Proof.

Fill in proof. □

Theorem 2.7. *Let $f : X \to Y$ be a map and $A \subseteq Y$. A is closed in $Y \Rightarrow f^{-1}(A)$ is closed in X.*

Proof.

Prove or disprove. Is the converse true? □

Definition A function $f : X \to Y$ is a *homeomorphism* $\Leftrightarrow$ f is invertible and both f and f^{-1} are maps.

Definition Two spaces X and Y are *homeomorphic* $\Leftrightarrow$ there exists a homeomorphism between them. We denote this by $X \approx Y$. This is the technical definition for 'X is the same as Y' discussed in Chapter 1.

Theorem 2.8. *Let X and Y be finite spaces. If $X \approx Y$ then ... use Theorem 2.2 to complete the theorem with something about the sizes of X and Y.*

Proof.

Prove the theorem. State the contrapositive of the theorem. Is it true? ☐

Example 2.9. Let $f : \mathbb{E} \to \mathbf{F^1}$ by $f(x) = x$. We really mean $f : \mathbb{R} \to \mathbb{R}$ where the first $\mathbb{R}$ has the $\mathbb{E}$ topology and the second has the $\mathbf{F^1}$ topology.

Is f continuous? Is f a homeomorphism? What can we conclude from this about whether $\mathbb{E} \approx \mathbf{F^1}$? (actually $\{\mathbb{R}, \mathbb{E}\} \approx \{\mathbb{R}, \mathbf{F^1}\}$).

Example 2.10. Let $f : \mathbf{H^1} \to \mathbb{E}$ by $f(x) = x$.

Is f continuous? Is f a homeomorphism? What can we conclude from this about whether $\mathbf{H^1} \approx \mathbb{E}$?

Example 2.11. Let $X = [0, 1)$ (with the subspace topology from $\mathbb{E}$), $Y = \mathbf{S}^1$ (with the subspace topology from $\mathbb{E}^2$) Define $f : X \to Y$ by $f(x) = (\cos(2\pi x), \sin(2\pi x))$.

Is f continuous? Is f a homeomorphism? What can we conclude from this about whether $[0, 1) \approx \mathbf{S}^1$?

Theorem 2.12. *Let $X = (0, 1)$ and $Y = (3, 5)$ (both with the subspace topology from $\mathbb{E}$), then $X \approx Y$.*

Proof.

Give all the details here. ☐

Theorem 2.13. *Let $X = \mathbb{E}$ and $Y = (0, 1)$ (with the subspace topology from $\mathbb{E}$), then $X \approx Y$.*

Proof.

Prove or disprove. ☐

Definition One of the fundamental problems in topology is to take a collection of spaces and decide which pairs are homeomorphic and which are not. We will call this process *classifying* a collection of spaces. As we have seen, there may be pairs that we are unable to distinguish with the current techniques. We begin the process of classifying the various topologies on $\mathbb{R}^1$ in the following theorems.

Theorem 2.14.

Make up a theorem about the relationship between $\mathbf{D^1}$ and the other topologies $\mathbb{E}, \mathbf{H^1}, \mathbf{T^1}, \mathbf{F^1}$ for $\mathbb{R}$.

Proof.

Prove your theorem. ☐

Theorem 2.15.

Make up a theorem about the relationship between $\mathbf{T^1}$ and the other topologies $\mathbb{E}, \mathbf{H^1}, \mathbf{D^1}, \mathbf{F^1}$ for $\mathbb{R}$.

Proof.

Prove your theorem. ☐