Topology

St. Mary's College of Maryland Topology Class

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1 Topological Spaces

In this chapter we define topological spaces sdfadfg and look at a number of examples. In addition, we define subspaces and construct corresponding examples.

Definition A *topological space* is a non-empty set X together with a collection $\mathcal{T}$ of subsets of X such that:

1. The empty set is in $\mathcal{T}$.
2. The set X itself is in $\mathcal{T}$.
3. Any *finite* intersection of sets in $\mathcal{T}$ is also in $\mathcal{T}$.
4. *Any* union of sets in $\mathcal{T}$ is also in $\mathcal{T}$.

or more formally,

1. $\emptyset \in \mathcal{T}$.
2. $X \in \mathcal{T}$.
3. $A_i \in \mathcal{T},\ i = 1 \cdots n \Rightarrow \bigcap\{A_i | i = 1 \cdots n\} \in \mathcal{T}$.
4. $A_\alpha \in \mathcal{T},\ \alpha \in \Gamma \Rightarrow \bigcup\{A_\alpha | \alpha \in \Gamma\} \in \mathcal{T}$.

Definition Given a topological space $\{X, \mathcal{T}\}$, $\mathcal{T}$ is called the *topology*, the elements of X are called *points* and the elements of $\mathcal{T}$ are called *open sets*.

Definition A subset B of a topological space X is *closed* $\Leftrightarrow$ $X \setminus B$ (the compliment of B) is open.

Lemma 1.1. *If X is a set and $\mathcal{T}$ a collection of subsets of X such that $\forall\ A_1, A_2 \in \mathcal{T}, A_1 \cap A_2 \in \mathcal{T}$, then axiom 3 is satisfied for $\mathcal{T}$.*

Proof.

Fill in proof □

Example 1.2.
Let $X = \{a, b\}$ and $\mathcal{T} = \{\emptyset, \{a\}, \{a, b\}\}$, then $\{X, \mathcal{T}\}$ is a topological space.

Fill in proof

Example 1.3. Let $X = \{a, b, c\}$ and $\mathcal{T} = \{\emptyset, \{a\}, \{b\}, \{a, b, c\}\}$, then $\{X, \mathcal{T}\}$ is a topological space.

Show that this is false, and then make $\mathcal{T}$ into a topology by adding the fewest possible sets to $\mathcal{T}$.

Remark 1.4. From here on we will use the word *space* to mean a topological space.

Definition (Intuitive version- we'll make it more rigorous later) Two spaces are *the same* if the points are just renamed. They are *different* if they are not the same.

Example 1.5. Let $X = \{a, b\}$, $\mathcal{T}_1 = \{\emptyset, \{a\}, \{a, b\}\}$, $\mathcal{T}_2 = \{\emptyset, \{b\}, \{a, b\}\}$, $\mathcal{T}_3 = \{\emptyset, \{a, b\}\}$.

Describe the relationships between these three topologies.

Example 1.6. The following are *all* the *different* topologies for $X = \{a, b, c\}$

Fill this in.

Theorem 1.7. *Let X be a non-empty set and $\mathcal{T} = \{\emptyset, X\}$. Then $\mathcal{T}$ is a topology for X.*

Proof.

Fill in proof. □

Definition We call the topology in the theorem above the *trivial topology* for X.

Theorem 1.8. *Let X be a non-empty set and $\mathcal{T} = \{$all subsets of $X\}$. Then $\mathcal{T}$ is a topology for X.*

Proof.

Fill in proof. □

Definition We call the topology in the theorem above the *discrete topology* for X.

Definition Let $\{X, \mathcal{T}_X\}$ be a space, and $\mathcal{B}$ a collection of subsets of X such that every element of $\mathcal{B}$ is in $\mathcal{T}_X$, and every element of $\mathcal{T}_X$ can be written as a union of members of $\mathcal{B}$. We call $\mathcal{B}$ a *basis for $\mathcal{T}_X$*, and $\mathcal{T}_X$ the *topology induced by $\mathcal{B}$*.

Theorem 1.9. *Let X be a non-empty set and $\mathcal{B}$ a collection of subsets of X satisfying:*

1. *The union of all elements of $\mathcal{B}$ is X.*

2. *The intersection of any pair of elements of $\mathcal{B}$ is the union of elements of $\mathcal{B}$.*

Let $\mathcal{S} = \{$the set of all possible unions of elements of $\mathcal{B}\}$. Then $\mathcal{S}$ is a topology for X with basis $\mathcal{B}$.

Proof. Fill in proof. □

Definition Let $\mathbb{R}$ be the set of real numbers, $x \in \mathbb{R}$, and $\epsilon > 0$ then

$$N(x, \epsilon) = \{y \in \mathbb{R} | x - \epsilon < y < x + \epsilon\}.$$

Note that $N(x, \epsilon)$ can also be written as $\{y \in \mathbb{R} || x - y| < \epsilon\}$. We call $N(x, \epsilon)$ an *ϵ-ball about x.*

Theorem 1.10. *Let $X = \mathbb{R}$ and $\mathcal{B} = \{N(x, \epsilon) | x \in \mathbb{R}, \epsilon > 0\}$ then $\mathcal{B}$ is a basis for a topology on $\mathbb{R}$.*

Proof.
Fill in proof. Hint: Use the theorem above. □

Definition
Extend this idea to to $\mathbb{R}^n$ by defining $N(\mathbf{x}, \epsilon) \subset \mathbb{R}^{\mathbf{n}} = ...$

Theorem 1.11. *Let $X = \mathbb{R}^n$ and $\mathcal{B} = \{N(\mathbf{x}, \epsilon) | \mathbf{x} \in \mathbb{R}^n, \epsilon > 0\}$ then $\mathcal{B}$ is a basis for a topology on $\mathbb{R}^n$.*

Proof.
Fill in proof. Hint: Use the theorem above. □

Definition
With $\mathcal{B}$ defined as above we define $\mathbb{E}^n$ to be the topology induced by $\mathcal{B}$ and refer to $\mathbb{E}^n$ as the *Euclidean topology* for $\mathbb{R}^n$. We sometimes just refer to $\mathbb{E}^n$ instead of $\{\mathbb{R}^n, \mathbb{E}^n\}$ for the whole space.

Theorem 1.12.

$$\mathbb{E}^n = \{A \subseteq \mathbb{R}^n \ |\forall \mathbf{x} \in A, \exists \ \epsilon > 0 \text{ such that } N(\mathbf{x}, \epsilon) \subseteq A\}.$$

Proof.
Fill in proof □

Definition Define $\tilde{N}(x, \epsilon) = \{y \in \mathbb{R} | x - \epsilon < y \leq x\}$.

Theorem 1.13. *Let $X = \mathbb{R}$ and $\tilde{\mathcal{B}} = \{\tilde{N}(x, \epsilon) | x \in \mathbb{R}, \epsilon > 0\}$ then $\tilde{\mathcal{B}}$ is a basis for a topology on $\mathbb{R}$.*

Proof.

Fill in proof. Use the theorem above.. ☐

Definition We will call $\mathbf{H^1}$ the *half-open* topology for $\mathbb{R}$.

Theorem 1.14. *Let* $\mathbf{F^1} = \{A \subseteq \mathbb{R} | \mathbb{R} \setminus A$ *is a finite set*$\} \cup \emptyset$. *Then the pair* $\{\mathbb{R}, \mathbf{F^1}\}$ *is a topological space.*

Proof.

Fill in proof ☐

Definition We call $\mathbf{F^1}$ the *Finite-Complement Topology* for $\mathbb{R}$.

Definition Denote $\mathbb{R}$ with the trivial topology as $\{\mathbb{R}, \mathbf{T^1}\}$, and $\mathbb{R}$ with the discrete topology as $\{\mathbb{R}, \mathbf{D^1}\}$.

Example 1.15.

Determine whether sets consisting of single points are open in any of the five topologies $\mathbb{E}, \mathbf{H^1}, \mathbf{F^1}, \mathbf{T^1}$ and $\mathbf{D^1}$.

Definition Let X be a space, $A \subseteq X$ and $x \in X$. Then, x is a *limit point of* A $\Leftrightarrow$ ($\forall O$ open in X such that $x \in O$)$(O \cap (A \setminus \{x\}) \neq \emptyset)$.
$Cl(A)$ (the *closure of* A) $= A \cup \{$limit points of $A\}$.

Definition Let X be a space, $A \subseteq X$ and $x \in X$. Then, x is an *interior point of* A $\Leftrightarrow$ ($\exists O$ open in X such that $x \in O \subseteq A$),
$Int(A)$ (the *interior of* A) $= \{$interior points of $A\}$.

Example 1.16. Let $X = \{p, q, r\}, \mathcal{T} = \{\emptyset, \{r\}, \{p, q\}, \{p, q, r\}\}$ and $A = \{p, r\}$.

Determine if any of the points of X are limit points and/or interior points of A.

Example 1.17. Let $X = \mathbb{E}^2$ (really, $X = \mathbb{R}^2, \mathcal{T}_X = \mathbb{E}^2$) and $A = \{(x, y) | 0 \leq x < 2, 0 \leq y < 2\}$.

Determine if any of the points $(0,0), (1,1), (1,2), (3,3)$ of X are limit points and/or interior points of A.

Theorem 1.18. *In any space* X*, a set* $A \subseteq X$ *is open* $\Leftrightarrow$ *All points of* A *are interior.*

Proof.

Fill in proof. ☐

Theorem 1.19. *In any space* X*, a set* $A \subseteq X$ *is closed* $\Leftrightarrow$ A *contains all of its limit points.*

Proof.

Fill in proof. ☐

Definition Let X be a space with topology $\mathcal{T}_X$, and $Y \subseteq X$ where $Y \neq \emptyset$. Let $\mathcal{T}_Y = \{A \subseteq Y | \exists\, O$ open in $X(ie.O \in \mathcal{T}_X)$ with $A = O \cap Y\}$.

Theorem 1.20. *The collection $\mathcal{T}_Y$ is a topology for Y.*

Proof.

Fill in proof ☐

Definition With this topology, we call Y a *subspace* of X and we call $\mathcal{T}_Y$ the *subspace topology* for Y.

Example 1.21. Let $X = \{a, b, c, d\}, \mathcal{T}_X = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, b, c, d\}\}$, and $Y = \{a, b, d\}$. Give Y the subspace topology.

List the open sets in Y
Give examples of non-empty subsets of Y that are:

1. open in Y and open in X.
2. open in Y but not open in X.
3. open in X but not open in Y.

Example 1.22. Let $X = \mathbb{E}^2$ and $Y = \{(x, y)|0 \leq x < 1, 0 \leq y < 1\}$. Give Y the subspace topology.

Give examples of non-empty subsets of Y that are:

1. open in Y and open in X.
2. open in Y but not open in X.
3. open in X but not open in Y.

Theorem 1.23.

Make up a simple theorem about open subsets in subspaces.

Proof.

Fill in proof ☐

Definition Let $X = \mathbb{E}^n$ and $Y = \{(x_1, x_2, \cdots, x_n)|x_1^2 + x_2^2 + \cdots + x_n^2 = 1\}$. Give Y the subspace topology. We will refer to Y as $\mathbf{S}^{n-1}$.
If we change the '=' to '$\leq$', the resulting space is called $\mathbf{B}^n$.

Example 1.24.

Draw pictures of $\mathbf{S}^{n-1}$ and $\mathbf{B}^n$ for $n = 1, 2, 3$. In your pictures draw some representative open sets. Can you guess why we use $\mathbf{S}^{n-1}$ instead of $\mathbf{S}^n$?