# **Brandt Kronholm** – *Research Statement*

This statement gives a quick introduction to my research and outlines current and future projects including undergraduate and graduate research.

My area of research is integer partitions. The partitions of an integer $n$ are the ways that positive whole numbers (the parts of the partition), regardless of their order, can be summed to $n$. The five partitions of 4 are 4, $3+1$, $2+2$, $2+1+1$, and $1+1+1+1$ and we write $p(4)=5$. Partitions can be investigated by students from grade school to graduate school allowing for limitless research opportunities at any level. I am interested in the application of generating functions along with combinatorial methods to solve counting problems as well as the $p$-divisibility of $q$-series. Applications of partitions and $q$-series arise frequently in physics, computer science, statistical mechanics, string theory, and vertex operator algebras. A generating function is a formal power series whose coefficients $a_n$ are a sequence $\{a_0, a_1, ...\}$.

Euler is responsible for laying the foundations of partition theory in the eighteenth century by introducing generating functions. He considered the the following infinite product of geometric series on the left side of (1) below. Multiplying and collecting like terms results in the generating function for the partitions of $n$ on the right hand side of (2).

$$\prod_{i=1}^{\infty} \frac{1}{1-q^i} = \left(1+q+q^2+q^3\cdots\right) \times \left(1+q^2+q^4+q^6+\cdots\right) \times \left(1+q^3+q^6+q^9+\cdots\right)\cdots \tag{1}$$

$$= 1+q+2q^2+3q^3+5q^4+7q^5+11q^6+15q^7+\cdots = \sum_{n=0}^{\infty} p(n)q^n \tag{2}$$

Mathematicians such as Sylvester [16], Hardy [4], Rademacher, Ramanujan [13], Dyson [2], Gauss, and Jacobi have made significant contributions to the subject. Andrews [1], Ono [12], and most recently Mahlburg [11] continue to make partitions an extremely popular area of research.

One particularly surprising aspect of partitions is their curious congruence properties. In 1919, Ramanujan [13] initiated this line of inquiry when he observed and proved the following divisibility statements.

$$p(5k+4) \equiv 0 \ (mod\ 5) \tag{3}$$

$$p(7k+5) \equiv 0 \ (mod\ 7) \tag{4}$$

$$p(11k+6) \equiv 0 \ (mod\ 11) \tag{5}$$

Ramanujan's proofs rely on careful manipulation of generating functions. In 2000, Ono [12] proved that there are partition congruences of this nature for every prime $> 3$.

I study the restricted partition function $p(n,m)$ which enumerates the number of partitions of $n$ into exactly $m$ parts. The relationship between $p(n)$ and $p(n,m)$ is clear:

$$p(n) = p(n,1) + p(n,2) + p(n,3) + ... + p(n,n-1) + p(n,n). \tag{6}$$

My research has led me to Ramanujan congruence properties of $p(n,m)$. The following are several examples of my recent results [8]. For $j \geq 0$:

$$p(54j,3) \equiv 0 \pmod{27} \tag{7}$$

$$p(60j-5,5) \equiv 0 \pmod{5} \tag{8}$$

$$p(2940j,7) \equiv 0 \pmod{49} \tag{9}$$

$$p(304920j-33,11) \equiv 0 \pmod{121} \tag{10}$$

$$p(360360j,13) \equiv 0 \pmod{13}. \tag{11}$$

Theorem 1 below is the general theorem describing examples (7), (8), (9), (10), and (11) above. It was gained by the manipulation of generating functions [8].

**Theorem 1** *For $\ell$ an odd prime, $j \geq 0$, and $0 \leq k \leq \frac{\ell-3}{2}$ and $\alpha \geq 1$,*

$$p(lcm(\ell) \cdot \ell^{\alpha-1} \cdot j - k\ell, \ell) \equiv 0 \pmod{\ell^\alpha},$$

*where $lcm(m)$ is the least common multiple of the numbers from 1 to $m$.*

## Research Projects

There are many research projects I would like to consider in both the immediate and far future. What follows are foremost among them.

### Generalized Ramanujan Congruences for $p(n, \ell)$ for parts $2 \leq m \leq \ell$ modulo prime powers.

This is a natural extension of [8] in the same way that [7] is an extension of [6]. From [8] we have that $p(2940j, 7) \equiv 0 \pmod{49}$. In this project it will be shown that much more is true, namely that

$$p(2940j, 7) \equiv p(2940j, 6) \equiv p(2940j, 5) \equiv p(2940j, 4) \equiv p(2940j, 3) \equiv p(2940j, 2) \equiv 0 \pmod{49}. \tag{12}$$

This result reveals several new prime power divisibilities of the generating functions and has immediate consequences within combinatorial methods for the study of integer partitions.

### Piece-Wise Polynomials

The results I have were not achieved by examining generating functions alone. They were first conjectures made from a great collection of data that I created by exploiting classical combinatorial methods within integer partitions. Conveniently, this data can be generated by a very simple computer program. The data is in fact polynomial expressions for restricted partition functions which are called "piece-wise polynomials." Once created, these piece-wise polynomials become the fuel for research at all levels. For example, the partitions of $n$ into exactly 3 parts are fully characterized in the following piece-wise polynomial. It should be said that the following is new information.

$$\begin{aligned}
p(6k, 3) &= 3k^2 && (13)\\
p(6k+1, 3) &= 3k^2 + k && (14)\\
p(6k+2, 3) &= 3k^2 + 2k && (15)\\
p(6k+3, 3) &= 3k^2 + 3k + 1 && (16)\\
p(6k+4, 3) &= 3k^2 + 4k + 1 && (17)\\
p(6k+5, 3) &= 3k^2 + 5k + 2 && (18)
\end{aligned}$$

From this collection of polynomials it is possible to create the piece-wise polynomial for $p(n, 4)$. As of yet there are no general theorems - or even conjectures - regarding the partitions of $n$ into anything other than a prime number of parts.

For example, for $k \geq 0$, (19) and (20) below are the polynomial expression for the number of partitions of $60k + 15$ into exactly five parts and $60k + 9$ into exactly six parts.

$$p(60k+15, 5) = 4500\,k^4 + 5250\,k^3 + \frac{4525}{2}\,k^2 + \frac{855}{2}\,k + 30 \tag{19}$$

$$p(60k+9, 6) = 9000\,k^5 + 10125\,k^4 + \frac{13100}{3}\,k^3 + \frac{1785}{2}\,k^2 + \frac{515}{6}\,k + 3. \tag{20}$$

Though it is clear that $p(60k + 15, 5) \equiv 0 \pmod 5$, my published results do not capture this congruence and the only conjecture describing it with any sort of generalization requires further examination. These families of polynomials eagerly await exploration.

**Prime Power Congruences for $p(n,m)$**

Ramanujan attempted to generalize the congruences he discovered for products of powers of $5, 7$, and 11. His most general conjecture was slightly modified by G. N. Watson in 1938 and was completed by A. O. L. Atkin in 1967. Though I have answered this question in part in [8], there are "pathological" occurrences of prime power divisibilities. This path of research is to generalize the following example.

$$p(27720k + 638, 11) \equiv p(27720k + 1288, 11) \equiv p(27720k + 1948, 11) \equiv \cdots \equiv 0 \ (mod\ 11^2) \tag{21}$$

**Other Partition Functions in terms of $p(n,m)$**

The function $q(n,m)$ which partitions the number of partitions of $n$ into exactly $m$ unequal parts can be expressed in terms of $p(n,m)$ in the following way:

$$q(n,m) = p(n - \frac{m^2 - m}{2}, m) \tag{22}$$

A lengthy list of such interconnected partitions is found in [3] and elsewhere in the literature. It would be very fruitful to create and explore the Piece-Wise Polynomials and Ramanujan congruences of such partitions.

Given the information in lines (13) through (18) we quickly obtain the following example:

$$q(6k+3,3) = 3k^2 \tag{23}$$
$$q(6k+4,3) = 3k^2 + k \tag{24}$$
$$q(6k+5,3) = 3k^2 + 2k \tag{25}$$
$$q(6k+6,3) = 3k^2 + 3k + 1 \tag{26}$$
$$q(6k+7,3) = 3k^2 + 4k + 1 \tag{27}$$
$$q(6k+8,3) = 3k^2 + 5k + 2 \tag{28}$$

Though this example is elementary, the usefulness of piece-wise polynomial characterizations of different restricted partition functions is evident.

**The Algebra of Partition Functions**

A paper by Rodseth and Sellers [14] makes use of my techniques and some algebra to capture results concerning congruence properties for a general family of restricted partition functions. There are ways in which the results of this paper can be extended, namely the form for the number $n$ being partitioned can be tightened and made smaller making the results more interesting.

Papers by Nijenhuis and Wilf [17] and Kwong [9], [10] abut my results in [7] using theorems from algebra. There are many more results to be had here using a growing variety of techniques. I am interested in using their algebraic techniques to further expand my results.

# References

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