GENERALIZED CONGRUENCE PROPERTIES OF THE RESTRICTED PARTITION FUNCTION $P(N, M)$

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Abstract. Ramanujan-type congruences for the unrestricted partition function $p(n)$ are well known and have been studied in great detail. The existence of Ramanujan-type congruences are virtually unknown for $p(n, m)$, the closely related restricted partition function that enumerates the number of partitions of n into exactly m parts. Let ℓ be any odd prime. In this paper we establish explicit Ramanujan-type congruences for $p(n, \ell)$ modulo any power of that prime ℓ^α. In addition, we establish general congruence relations for $p(n, \ell)$ modulo ℓ^α for any n.

1. Introduction and a Result

A partition of a non-negative integer n is a non-increasing finite sequence of positive integers $\lambda_1, \lambda_2, ..., \lambda_r$ whose sum is n. The λ_i are called the parts of the partition. Let $p(n)$ denote the number of partitions of n. We agree that $p(0) = 1$ and that $p(n) = 0$ for $n \notin \mathbb{Z}_{\geq 0}$.

For example, there are five partitions of 4. One very natural way to list these partitions is by increasing number of parts:

$$4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1. \tag{1.1}$$

Hence $p(4) = 5$. The restricted partition function which enumerates the number of partitions of the non-negative integer n into exactly m parts is denoted by $p(n, m)$. From (1.1) we can see that there are two partitions of 4 into exactly two parts, namely $3+1$ and $2+2$ and we write $p(4, 2) = 2$. It is clear that $p(n, m)$ is closely related to the unrestricted partition function $p(n)$ in that

$$p(n) = p(n, 1) + p(n, 2) + p(n, 3) + ... + p(n, n-1) + p(n, n). \tag{1.2}$$

where for $n < m$, $p(n, m) = 0$.

In 1919, Ramanujan [15] discovered three surprising congruences for $p(n)$:

$$p(5j + 4) \equiv 0 \pmod 5,$$

$$p(7j + 5) \equiv 0 \pmod 7,$$

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and

$$p(11j+6) \equiv 0 \pmod{11}.$$

Though he was able to provide proofs for these congruences, he felt there was something unusual about them [14].

> *It appears that there are no equally simple properties for any moduli involving primes other than these three (i.e. $m = 5, 7, 11$).*

His intuition was correct. It took nearly fifty years after Ramanujan's initial discovery for A. L. O. Atkin [3] to discover a fourth such congruence:

$$p(206839j+2623) \equiv 0 \pmod{17}.$$

More than thirty years would pass before Ken Ono [13] proved for primes $\ell \geq 5$ that there are infinitly many Ramanujan-like congruences of the form

$$p(Aj+B) \equiv 0 \pmod{\ell}. \tag{1.3}$$

A similar time scale follows theorems regarding congruences modulo prime powers and composite munbers. In 1919 Ramanujan made a conjecture which was eventually refined by Watson in 1938 [17] and proved by Atkin in 1967 [4].

Theorem 1. [17] *If $24n-1 \equiv 0$ (mod $5^a 7^b 11^c$) then $p(n) \equiv 0$ (mod $5^a 7^d 11^c$) where $a, b, c \geq 0$ and $d = \left[\frac{b+2}{2}\right]$ if $b > 0$, $d = 0$ if $b = 0$.*

In 2000, Scott Ahlgren [1] extended the results of Ono [13] by showing that the prime ℓ may in fact be replaced by an arbitrary prime power ℓ^α extending (1.3) to

$$p(Aj+B) \equiv 0 \pmod{\ell^\alpha}. \tag{1.4}$$

For their work on the Theory of Invariants, Sylvester [16] (along with Cayley) did an enrmous amount of research on closed term formulae for $p(n,m)$ at the end of the 19^{th} century. Mathematicians including Erdos and Lehner, Gupta, Gwyther and Miller [6], Chowla, Szekeres, and Haselgrove and Templerly have made great contributions to the behavior of $p(n,m)$ [5]. A paper by Wilf and Nijenhuis [12] describes the periodicity of $p(n,m)$ modulo a prime. This useful result was followed by three papers by Kwong [9, 10, 11] extending the results of [12] to prime powers. Despite an abundance of material on the function $p(n,m)$ and the close relationship between $p(n)$ and $p(n,m)$, there are no results regarding Ramanujan-type congruences for $p(n,m)$ save those by the author [7, 8].

The purpose of this paper is to establish explicit Ramanujan-type congruences as in (1.4) for $p(n, m)$ modulo arbitrary powers of the prime ℓ with $\ell \geq 3$. For $j \geq 0$:

$$p(54j, 3) \equiv 0 \pmod{27}$$
$$p(60j - 5, 5) \equiv 0 \pmod{5}$$
$$p(2940j, 7) \equiv 0 \pmod{49}$$
$$p(304920j - 33, 11) \equiv 0 \pmod{121}$$
$$p(360360j, 13) \equiv 0 \pmod{13}.$$

Our first result is stated as

Corollary 1. *For ℓ an odd prime, $j \geq 0$, and $0 \leq k \leq \frac{\ell-3}{2}$ and $\alpha \geq 1$,*

$$p(lcm(\ell) \cdot \ell^{\alpha-1} \cdot j - k\ell, \ell) \equiv 0 \pmod{\ell^\alpha},$$

where $lcm(m)$ is the least common multiple of the numbers from 1 to m.

Corollary 1 gives us $\frac{\ell-1}{2}$ Ramanujan-like congruences for each ℓ. Corollary 1 was previously known only for the case $\alpha = 1$ [7].

Example 1. *Examples of Corollary 1 modulo powers of 3.*

$$p(6j, 3) \equiv 0 \pmod{3}, \qquad p(18j, 3) \equiv 0 \pmod{9}$$
$$p(54j, 3) \equiv 0 \pmod{27}, \qquad p(162j, 3) \equiv 0 \pmod{81}$$

Example 2. *Examples of Corollary 1 modulo powers of 5.*

$$p(60j, 5) \equiv 0 \pmod{5}, \qquad p(60j - 5, 5) \equiv 0 \pmod{5}$$
$$p(300j, 5) \equiv 0 \pmod{25}, \qquad p(300j - 5, 5) \equiv 0 \pmod{25}$$
$$p(1500j, 5) \equiv 0 \pmod{125}, \qquad p(1500j - 5, 5) \equiv 0 \pmod{125}$$

2. Statement of Theorems and Examples

We require the following definition to state our theorems.

Definition 1. *Let m, h, j and r be integers such that $m > 0$, $j \geq 0$, and $-\frac{m^2-3m}{2} \leq r < lcm(m) \cdot h$. Let $n = lcm(m) \cdot h \cdot j + r$ and $n' = lcm(m) \cdot h(j+1) - \frac{m^2-3m}{2} - r$.*

We will define $p(n, m)$ and $p(n', m)$ to be palindromic partners.

Example 3. $p(12j + 1, 4)$ *and* $p(12j + 9, 4)$ *are palindromic partners.*

We now state our main theorems. Though our theorems are stated for odd primes, it should be said that since $p(n, 2) = \lfloor \frac{n}{2} \rfloor$ somewhat similar, but by no means identical, results for the even prime 2 are easily formulated.

Theorem 2. *Let ℓ be an odd prime, $n = lcm(\ell) \cdot \ell^{\alpha-1} \cdot j + r$ and n' be its palindromic partner so that $n' = lcm(\ell) \cdot \ell^{\alpha-1}(j+1) - \frac{\ell^2-3\ell}{2} - r$. For k an integer such that $min\{n, n-k\ell, n', n'+k\ell\} \geq -\frac{\ell^2-3\ell}{2}$ and $\alpha \geq 1$, then*

$$p(n,\ell) - p(n',\ell) + p(n'+k\ell,\ell) - p(n-k\ell,\ell) \equiv 0 \pmod{\ell^\alpha}. \tag{2.1}$$

Note that not only are $p(n,\ell)$ and $p(n',\ell)$ palindromic partners, but also that $p(n'+k\ell,\ell)$ and $p(n-k\ell,\ell)$ are as well.

In the following examples of Theorem 2, we consider $p(n,5)$ modulo 5^α for $n = 47$. We choose $n = 47$ simply for the purposes of the following examples. Examples 4, 5, 6, 7, 8, and 9 highlight the variable k. Examples 10 and 11 have $k = 1$ and highlight the variable α.

Example 4. $\alpha = 1$, $k = 1$,

$$p(47,5)-p(8,5)+p(13,5)-p(42,5) = 2062-3+18-1342 = 735 \equiv 0 \pmod 5$$

Example 5. $\alpha = 1$, $k = 2$,

$$p(47,5)-p(8,5)+p(18,5)-p(37,5) = 2062-3+57-831 = 1285 \equiv 0 \pmod 5$$

Example 6. $\alpha = 1$, $k = 3$,

$$p(47,5)-p(8,5)+p(23,5)-p(32,5) = 2062-3+141-480 = 1720 \equiv 0 \pmod 5$$

Example 7. $\alpha = 1$, $k = 4$,

$$p(47,5)-p(8,5)+p(28,5)-p(27,5) = 2062-3+291-255 = 2095 \equiv 0 \pmod 5$$

Example 8. $\alpha = 1$, $k = 5$,

$$p(47,5)-p(8,5)+p(33,5)-p(22,5) = 2062-3+540-119 = 2480 \equiv 0 \pmod 5$$

Example 9. $\alpha = 1$, $k = 6$,

$$p(47,5)-p(8,5)+p(38,5)-p(17,5) = 2062-3+918-47 = 2930 \equiv 0 \pmod 5$$

Example 10. $\alpha = 2$, $k = 1$, $n = 47 = lcm(5) \cdot 5 \cdot 0 + 47$ *so* $n' = lcm(5) \cdot 5 \cdot 1 - 5 - 47 = 248$,

$$p(47,5) - p(248,5) + p(253,5) - p(42,5)$$
$$= 2062 - 1366617 + 1479072 - 1342 = 113175 \equiv 0 \pmod{25}$$

Example 11. $\alpha = 3$, $k = 1$, $n = 47 = lcm(5) \cdot 5^2 \cdot 0 + 47$ *so* $n' = lcm(5) \cdot 5^2 \cdot 1 - 5 - 47 = 1448$,

$$p(47,5) - p(1448,5) + p(1453,5) - p(42,5)$$
$$= 2062 - 1493467461 + 1494035616 - 1342 = 568875 \equiv 0 \pmod{125}$$

Our next result is a special case of Theorem 2. When n is a multiple of the prime ℓ, the congruence relation resides in the difference of a single case of palindromic partners and not the sum of the differences of two instances of palindromic partners as in Theorem 2. For example, Theorem 2 tells us that

$$p(35,5) - p(20,5) + p(25,5) - p(30,5) \equiv 0 \pmod 5 \tag{2.2}$$

However, it is true that

$$p(35,5) - p(20,5) \equiv 0 \pmod 5. \tag{2.3}$$

Theorem 3 captures the generalization of this congruence.

Theorem 3. *Let ℓ be an odd prime, $n = lcm(\ell) \cdot \ell^{\alpha-1} \cdot j + \ell k$ and n' be its palindromic partner so that $n' = lcm(\ell) \cdot \ell^{\alpha-1} \cdot (j+1) - \frac{\ell^2-3\ell}{2} - \ell k$. For $min\{n, n'\} \geq -\frac{\ell^2-3\ell}{2}$ and $\alpha \geq 1$ we have*

$$p(n,\ell) - p(n',\ell) \equiv 0 \pmod{\ell^\alpha}. \tag{2.4}$$

We illustrate Theorem 3 with the following examples.

Example 12. *Let $\ell = 5$, $\alpha = 1$ so that $n = lcm(5) + 5k$ there are only $lcm(5)/(5 \cdot 2) = 6$ palindromic partners whose difference is congruent to zero modulo 5.*

$$p(60j+55,5) - p(60j,5) \equiv 0 \pmod 5 \tag{2.5}$$

$$p(60j+50,5) - p(60j+5,5) \equiv 0 \pmod 5 \tag{2.6}$$

$$p(60j+45,5) - p(60j+10,5) \equiv 0 \pmod 5 \tag{2.7}$$

$$p(60j+40,5) - p(60j+15,5) \equiv 0 \pmod 5 \tag{2.8}$$

$$p(60j+35,5) - p(60j+20,5) \equiv 0 \pmod 5 \tag{2.9}$$

$$p(60j+30,5) - p(60j+25,5) \equiv 0 \pmod 5 \tag{2.10}$$

Remark 1. *Compare (2.9) to (2.3).*

Example 13. *With $\ell = 5$, $\alpha = 2$ so that $n = lcm(5) \cdot 5 + 5k$ there are now $5 \cdot lcm(5)/(5 \cdot 2) = 30$ palindromic partners whose difference is congruent to zero modulo 5.*

$$p(300j+295,5) - p(300j,5) \equiv 0 \pmod{25}$$
$$p(300j+290,5) - p(300j+5,5) \equiv 0 \pmod{25}$$
$$p(300j+285,5) - p(300j+10,5) \equiv 0 \pmod{25}$$
$$p(300j+280,5) - p(300j+15,5) \equiv 0 \pmod{25}$$

$$p(300j + 275, 5) - p(300j + 20, 5) \equiv 0 \pmod{25}$$

$$\vdots$$

Corollary 1 is proved from Theorem 3. The proof appears in section 5.

3. Background

The generating functions for both $p(n)$ and $p(n, m)$ are similar.

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{k=1}^{\infty} \frac{1}{(1-q^k)} = \frac{1}{(q;q)_\infty} \tag{3.1}$$

$$\sum_{n=0}^{\infty} p(n,m)q^n = q^m \cdot \prod_{k=1}^{m} \frac{1}{(1-q^k)} = \frac{q^m}{(q;q)_m} \tag{3.2}$$

As in (1.2), we relate the generating functions in (3.1) and (3.2) by (3.3).

$$\frac{1}{(q;q)_\infty} = 1 + \frac{q}{1-q} + \frac{q^2}{(q;q)_2} + \frac{q^3}{(q;q)_3} + \frac{q^4}{(q;q)_4} + \frac{q^5}{(q;q)_5} + \cdots \tag{3.3}$$

Many partition theorems are proved by studying their generating functions. Our results will come from congruentially identifying generating functions like (3.2) to polynomials and then investigating these polynomials with techniques both old and new.

We require the following information to prove our results.

Lemma 1. *For $\beta \geq 0$ and $n \geq \beta m$*

$$p(n,m) - p\left(n - \beta m, m\right) = \sum_{i=0}^{\beta-1} p(n-1-im, m-1) \tag{3.4}$$

Proof of Lemma 1 Let $\beta \geq 0$ and $n \geq \beta m$

$$\sum_{n \geq 0} \left[p(n,m) - p(n-\beta m, m)\right] q^n = \frac{q^m(1-q^{\beta m})}{(q;q)_m} \tag{3.5}$$

$$= \frac{q^{m-1}}{(q;q)_{m-1}} \cdot \frac{q(1-q^{\beta m})}{(1-q^m)} \tag{3.6}$$

$$= \sum_{k \geq 0} p(k, m-1)q^k \cdot \sum_{i=0}^{\beta-1} q^{im+1} \tag{3.7}$$

And so

$$p(n,m) - p\left(n - \beta m, m\right) = \sum_{i=0}^{\beta-1} p(n-1-im, m-1). \quad \square \tag{3.8}$$

For our purposes we consider two cases of Lemma 1 into the following remarks as β is either odd or even.

Remark 2. *For β as in Lemma 1 with $\beta = 2b + 1$,*

$$p(n,m) - p\left(n - (2b+1)m, m\right) = \sum_{i=-b}^{b} p(n-1-(b+i)m, m-1). \tag{3.9}$$

For β as in Lemma 1 with $\beta = 2b$,

$$p(n,m) - p\left(n - (2b)m, m\right) = \sum_{i=-b}^{b-1} p(n-1-(b+i)m, m-1). \tag{3.10}$$

We observe that (3.9) and (3.10) are a result of reindexing the sum and adjusting the summand accordingly.

Definition 2. [7] *A polynomial $P(q) = a_0 + a_1 q + \cdots + a_n q^n$ of degree n is called anti-reciprocal if for each i,*

$$a_i = -a_{n-i} \tag{3.11}$$

or if

$$q^n \left(P(1/q)\right) = -P(q). \tag{3.12}$$

When $P(q)$ is an antireciprocal polynomial we will call the coefficients a_i and a_{n-i} antireciprocal partners.

Though the following definition of reciprocal polynomial is not directly used in any proof in this paper, we include it here so that we may fully describe the methods of proof.

Definition 3. [2] *A polynomial $P(q) = a_0 + a_1 q + \cdots + a_n q^n$ of degree n is called reciprocal if for each i,*

$$a_i = a_{n-i} \tag{3.13}$$

or if

$$q^n \left(P(1/q)\right) = P(q). \tag{3.14}$$

When $P(q)$ is an reciprocal polynomial we will call the coefficients a_i and a_{n-i} reciprocal partners.

The following lemma is crucial for our results. It was first described by Albert Nijenhuis and Herb Wilf [12] and later by the author [8] but only for the case modulo ℓ. The extension to prime powers ℓ^α is due to Y. H. Kwong who was a student of Nijenhuis'. The following lemma used for the results in this paper is the end result of a series of four papers beginning with [12] and including [11],[9], and [10].

Lemma 2. [10] *For a nonnegative integer t, the sequence $\{p(n,t) \pmod{\ell^\alpha}\}_{n\geq 0}$ is periodic with minimum period $lcm(\ell)\cdot \ell^{\alpha-1}$ so long as*

$$\sum_{\delta\geq 0} \phi(\ell^\delta)\left\lfloor \frac{t}{\ell^\delta}\right\rfloor \leq \ell,$$

where ϕ is Euler's function.

Remark 3. *Given a nonnegative integer t, let d be any of the natural numbers that are multiples of $lcm(t)$ and let ℓ^α be a primary factor of d. Set $K(\ell,t,d) = \sum_{\delta\geq 0}\phi(\ell^\delta)\lfloor \frac{t}{\ell^\delta}\rfloor$. Whenever $K(\ell,t,d)\leq \ell$ we will say that Kwong's Criterion is satisfied. Moreover, when $t=\ell-1$ and $d = lcm(\ell)\cdot\ell^{\alpha-1}\cdot j$, for $j\geq 0$ we have*

$$K(\ell,\ell-1,lcm(\ell)\cdot\ell^{\alpha-1}\cdot j) = \ell-1<\ell, \tag{3.15}$$

and Kwong's Criterion is satisfied.

Kwong's Criterion provides us with an easy way to determine if a rational function of the form

$$\frac{(1-q^d)}{(q;q)_t} \tag{3.16}$$

can be congruentially identified to a polynomial. We exploit certain properties of the resulting polynomial to gain our results.

Lemma 3. *Let ℓ,t,d be as in Remark 3 so that Kwong's Criterion is satisfied. Let $n\geq d-\binom{t}{2}$. The generating function for the difference between the number of partitions of n into exactly t parts and the number of partitions of $n-d$ into exactly t parts is congruent modulo ℓ^α to a polynomial. We will call this polynomial $A(q;\ell,t,d)$.*

$$\sum_{n=0}^{\infty}\left(p(n,t)-p(n-d,t)\right)q^n = \frac{q^t(1-q^d)}{(q;q)_t} \equiv A(q;\ell,t,d) \pmod{\ell^\alpha} \tag{3.17}$$

Depending on $t \pmod 4$, $A(q;\ell,t,d)$ has the following properties:

- *If $t\equiv 0 \pmod 4$, then $A(q;\ell,t,d)$ is an antireciprocal polynomial of degree $d-\frac{t^2-3t}{4}$ with $d+1$ terms.*
- *If $t\equiv 1 \pmod 4$, then $A(q;\ell,t,d)$ is a reciprocal polynomial of degree $d-\frac{t^2-3t+2}{4}$ with d terms.*
- *If $t\equiv 2 \pmod 4$, then $A(q;\ell,t,d)$ is an antireciprocal polynomial of degree $d-\frac{t^2-3t+2}{4}$ with d terms.*
- *If $t\equiv 3 \pmod 4$, then $A(q;\ell,t,d)$ is a reciprocal polynomial of degree $d-\frac{t^2-3t}{4}$ with $d+1$ terms.*

We note here that the degree of the polynomial $A(q;\ell,t,d)$ is larger than that specified by Definiton 2 and Definition 3. This is due to including a certain number of terms with coefficient zero. We include these terms so that we may make efficient use of the properties of reciprocal and anti-reciprocal polynomials. Given that we write a polynomial in canonical form, these properties are unaltered by including terms with coefficient zero on the leading side of the polynomial and an equivalent number of likewise terms on the opposite end of the polynomial. These additional terms are obtained in the following way: Since $\frac{(1-q^d)}{(q;q)_t}$ is purely periodic modulo ℓ^α with period d by Lemma 2 then it identifies to a polynomial with d terms. Moreover, it is easy to show that $A(q;\ell,t,d)$ is a polynomial of degree $d-\frac{t^2-3t}{2}$ by satisfying either Definition 2 or Definition 3. This difference between the number of terms d and the degree of the polynomial is where the additional terms with coefficient zero come from.

For example, in Lemma 3 letting $t=2$, $\ell=3$, and $d=lcm(3)\cdot 3=18$ we may identify the generating function for the difference between the number of partitions of n into exactly 2 parts and $n-18$ into exactly 2 parts in line (3.18) to the polynomial $A(q;3,2,18)$ in line (3.19).

$$\sum_{n=0}^{\infty}\left[p\left(n,2\right)-p(n-lcm(3)\cdot 3,2)\right]q^n=\frac{q^2(1-q^{18})}{(1-q)(1-q^2)} \tag{3.18}$$

$$\equiv 0q+q^2+q^3+2q^4+2q^5+3q^6+3q^7+4q^8+4q^9$$

$$-4q^{10}-4q^{11}-3q^{12}-3q^{13}-2q^{14}-2q^{15}-q^{16}-q^{17}-0q^{18} \pmod 9. \tag{3.19}$$

Indeed, (3.19) is an antireciprocal polynomial of degree 18 with 18 terms by Lemma 3.

Theorem 4 is the partition theoretic interpretation of Lemma 3.

Theorem 4. *Let ℓ, t, d be as in Remark (3) so that Kwong's Criterion is satisfied and $N\geq 0$, $M>N$.*

- *If $t\equiv 0 \pmod 4$, then*

$$p\left(dN-\left(\frac{t^2-3t}{4}\right)+i,t\right)\equiv -p\left(dM-\left(\frac{t^2-3t}{4}\right)-i,t\right) \pmod{\ell^\alpha}.$$

- *If $t\equiv 1\,(mod\,4)$, then*

$$p\left(dN-\left(\frac{t^2-3t-2}{4}\right)+i,t\right)\equiv p\left(dM-\left(\frac{t^2-3t+2}{4}\right)-i,t\right) \pmod{\ell^\alpha}.$$

- *If $t\equiv 2\,(mod\,4)$, then*

$$p\left(dN-\left(\frac{t^2-3t-2}{4}\right)+i,t\right)\equiv -p\left(dM-\left(\frac{t^2-3t+2}{4}\right)-i,t\right) \pmod{\ell^\alpha}.$$

- *If* $t \equiv 3\,(mod\,4)$, *then*

$$p\left(dN - \left(\frac{t^2-3t}{4}\right) + i, t\right) \equiv p\left(dM - \left(\frac{t^2-3t}{4}\right) - i, t\right) \pmod{\ell^\alpha}.$$

The importance of Theorem 4 is that depending on t and for fixed i, these partitions correspond to the reciprocal or antireciprocal partners of the polynomial $A(q;\ell,t,d)$ in (3.17). Moreover, the partitions in this theorem satisfy Definition 1 and are palindromic partners. Theorem 4 is a most general congruence relation for all n into exactly t parts.

Remark 4. *Let* ℓ, t, d *be so that Kwong's Criterion (Remark (3)) is satisfied and* $N \geq 0,\ M > N$.

- *If* $t \equiv 0 \pmod 4$, *then*

$$\sum_{i=0}^{d(M-N)} \left[p\left(dN - \left(\frac{t^2-3t}{4}\right) + i, t\right) + p\left(dM - \left(\frac{t^2-3t}{4}\right) - i, t\right)\right]$$

$$\equiv 0\ (mod\ \ell^\alpha). \tag{3.20}$$

- *If* $t \equiv 2 \pmod 4$, *then*

$$\sum_{i=0}^{d(M-N)} \left[p\left(dN - \left(\frac{t^2-3t-2}{4}\right) + i, t\right) + p\left(dM - \left(\frac{t^2-3t+2}{4}\right) - i, t\right)\right]$$

$$\equiv 0\ (mod\ \ell^\alpha). \tag{3.21}$$

4. Proofs of Theorems

For Theorem 2 we will prove the case for $\ell \equiv 1 \pmod 4$. We begin with line (3.20) in Remark 4 where Kwong's Criterion is satisfied.

$$\sum_{i=0}^{d(M-N)} \left[p\left(dN - \left(\frac{t^2-3t}{4}\right) + i, t\right) + p\left(dM - \left(\frac{t^2-3t}{4}\right) - i, t\right)\right]$$

$$\equiv 0 \pmod{\ell^\alpha}. \tag{4.1}$$

For $N = j \geq 0$ and $M = j+1$ we set $d = lcm(\ell)\ell^{\alpha-1}$ with $\ell \equiv 1 \pmod 4$, so that $t = \ell-1$. Set $i = r + \ell\left(\frac{\ell-1}{4} - k + i\right)$ and for $0 \leq k \leq lcm(\ell-1)$, reindex from $i=0$ to $k-1$. Note $K(\ell, \ell-1, lcm(\ell)\ell^{\alpha-1}j) = \ell - 1$ so that Kwong's Criterion remains satisfied. In essence we are manipulating the polynomial $A(q;\ell,\ell-1,lcm(\ell)\ell^{\alpha-1}j)$. Rewrite (4.1):

$$\sum_{i=0}^{k-1} \left[p\left(lcm(\ell)\ell^{\alpha-1}j - \left(\frac{\ell^2-5\ell+4}{4}\right) + r + \ell\left(\frac{\ell-1}{4} - k + i\right), \ell - 1\right)\right.$$

(4.2)

$$+p\left(lcm(\ell)\ell^{\alpha-1}(j+1)-\left(\frac{\ell^2-5\ell+4}{4}\right)-r-\ell\left(\frac{\ell-1}{4}-k+i\right),\ell-1\right)\Bigg]$$

$$\equiv\sum_{i=0}^{k-1}\Big[p\left(lcm(\ell)\ell^{\alpha-1}j+r-1-(k-1)\ell+i\ell,\ell-1\right)$$

$$+p\left(lcm(\ell)\ell^{\alpha-1}(j+1)-\left(\frac{\ell^2-3\ell}{2}\right)-r+k\ell-1-i\ell,\ell-1\right)\Bigg] \tag{4.3}$$

$$\equiv\sum_{i=0}^{k-1}p\left(lcm(\ell)\ell^{\alpha-1}j+r-1-i\ell,\ell-1\right)$$

$$+\sum_{i=0}^{k-1}p\left(lcm(\ell)\ell^{\alpha-1}(j+1)-\left(\frac{\ell^2-3\ell}{2}\right)-r+k\ell-1-i\ell,\ell-1\right)$$

$$\equiv 0 \pmod{\ell^\alpha} \tag{4.4}$$

Observe that $lcm(\ell)\ell^{\alpha-1}j+r$ and $lcm(\ell)\ell^{\alpha-1}(j+1)-\left(\frac{\ell^2-3\ell}{2}\right)-r$ are palindromic partners. Set $lcm(\ell)\ell^{\alpha-1}j+r=n$ so that $n'=lcm(\ell)\ell^{\alpha-1}(j+1)-\left(\frac{\ell^2-3\ell}{2}\right)-r$ and write

(4.5)

$$\sum_{i=0}^{k-1}p\left(n-1-i\ell,\ell-1\right)+\sum_{i=0}^{k-1}p\left(n'+k\ell-1-i\ell,\ell-1\right)\equiv 0 \pmod{\ell^\alpha}$$

By Lemma 1, (4.5) is equal to

$$[p(n,\ell)-p(n-k\ell,\ell)]+[p(n'+k\ell,\ell)-p(n',\ell)]\equiv 0 \pmod{\ell^\alpha} \tag{4.6}$$

$$=p(n,\ell)-p(n',\ell)+p(n'+k\ell,\ell)-p(n-k\ell,\ell)\equiv 0 \pmod{\ell^\alpha} \tag{4.7}$$

The case for $\ell\equiv 3 \pmod 4$ is done by the same method and is left for the reader. □

For Theorem 3 we will prove the case for $\ell\equiv 1 \pmod 4$ We start with line (3.20) in Remark 4 where Kwong's Criterion is satisfied.

$$\sum_{i=0}^{d(M-N)}\left[p\left(dN-\left(\frac{t^2-3t}{4}\right)+i,t\right)+p\left(dM-\left(\frac{t^2-3t}{4}\right)-i,t\right)\right]$$

$$\equiv 0 \pmod{\ell^\alpha}. \tag{4.8}$$

For $N=j\geq 0$ and $M=j+1$ we set $d=lcm(\ell)\ell^{\alpha-1}$ with $\ell\equiv 1 \pmod 4$, so that $t=\ell-1$. Set $i=i+\frac{lcm(\ell-1)}{2}+\frac{\ell-5}{4}$ and for $0\leq k\leq lcm(\ell-1)-1$, reindex from $i=0$ to $k-\frac{lcm(\ell-1)}{2}$. Note $K(\ell,\ell-1,lcm(\ell)\cdot\ell^{\alpha-1}j)=\ell-1$ so that Kwong's Criterion remains satisfied. Rewrite (4.8) as

$$\sum_{i=0}^{k-\frac{lcm(\ell-1)}{2}} \Big[p\left(lcm(\ell)\ell^{\alpha-1}j-\left(\tfrac{\ell^2-5\ell+4}{4}\right)+\ell\left(i+\tfrac{lcm(\ell-1)}{2}+\tfrac{\ell-5}{4}\right),\ell-1\right)$$
$$+\,p\left(lcm(\ell)\ell^{\alpha-1}(j+1)-\left(\tfrac{\ell^2-5\ell+4}{4}\right)-\ell\left(i+\tfrac{lcm(\ell-1)}{2}+\tfrac{\ell-5}{4}\right),\ell-1\right)\Big]$$
$$\equiv \sum_{i=0}^{k-\frac{lcm(\ell-1)}{2}+\frac{\ell-5}{4}} \Big[p\left(lcm(\ell)\ell^{\alpha-1}j-\left(\tfrac{\ell^2-5\ell+4}{4}\right)+\ell\left(i+\tfrac{lcm(\ell-1)}{2}\right),\ell-1\right)$$
$$+\,p\left(lcm(\ell)\ell^{\alpha-1}(j+1)-\left(\tfrac{\ell^2-5\ell+4}{4}\right)-\ell\left(i+\tfrac{lcm(\ell-1)}{2}\right),\ell-1\right)\Big]$$
$$\equiv 0 \pmod{\ell^\alpha} \tag{4.9}$$

Observe in 4.9 that for $i=0$ we have

$$p\left(lcm(\ell)\ell^{\alpha-1}j-\left(\frac{\ell^2-5\ell+4}{4}\right)+\ell\left(\frac{lcm(\ell-1)}{2}\right),\ell-1\right)$$
$$+p\left(lcm(\ell)\ell^{\alpha-1}(j+1)-\left(\frac{\ell^2-5\ell+4}{4}\right)-\ell\left(\frac{lcm(\ell-1)}{2}\right),\ell-1\right)$$
$$\equiv 0 \pmod{\ell^\alpha}. \tag{4.10}$$

Since

$$lcm(\ell)\ell^{\alpha-1}j+\frac{lcm(\ell)}{2}=lcm(\ell)\ell^{\alpha-1}(j+1)-\frac{lcm(\ell)}{2}$$

Then it follows that

$$p\left(lcm(\ell)\ell^{\alpha-1}(j+1)-\left(\frac{\ell^2-5\ell+4}{4}\right)-\ell\left(\frac{lcm(\ell-1)}{2}\right),\ell-1\right)$$
$$\equiv 0 \pmod{\ell^\alpha}. \tag{4.11}$$

We subtract the term in (4.11) from equation (4.9) and consider

$$\sum_{i=-\left(k-\frac{lcm(\ell-1)}{2}+\frac{\ell-5}{4}\right)}^{0} p\left(lcm(\ell)\ell^{\alpha-1}j+k\ell-1-\ell\left(k-\tfrac{lcm(\ell-1)}{2}+\tfrac{\ell-5}{4}+i\right),\ell-1\right)$$
$$+\sum_{i=1}^{k-\frac{lcm(\ell-1)}{2}+\frac{\ell-5}{4}} p\left(lcm(\ell)\ell^{\alpha-1}j+k\ell-1-\ell\left(k-\tfrac{lcm(\ell-1)}{2}+\tfrac{\ell-5}{4}+i\right),\ell-1\right)$$
$$\equiv \sum_{i=-\left(k-\frac{lcm(\ell-1)}{2}+\frac{\ell-5}{4}\right)}^{k-\frac{lcm(\ell-1)}{2}+\frac{\ell-5}{4}} p\left(lcm(\ell)\ell^{\alpha-1}j+k\ell-1-\ell\left(k-\tfrac{lcm(\ell-1)}{2}+\tfrac{\ell-5}{4}+i\right),\ell-1\right)$$
$$\equiv \sum_{i=0}^{2k-lcm(\ell-1)+\frac{\ell-3}{2}-1} p\left(lcm(\ell)\ell^{\alpha-1}j+k\ell-1-\ell i,\ell-1\right)\equiv 0 \pmod{\ell^\alpha} \tag{4.12}$$

By Lemma 1 (4.12) is equal to

$$p(lcm(\ell)\ell^{\alpha-1}j+k\ell,\ell)$$

$$-p\left(lcm(\ell)\ell^{\alpha-1}j + k\ell - \left(2k - lcm(\ell-1) + \tfrac{\ell-3}{2}\right)\ell, \ell\right)$$

$$\equiv p(lcm(\ell)\ell^{\alpha-1}j + k\ell, \ell) - p\left(lcm(\ell)\ell^{\alpha-1}(j+1) - \frac{\ell^2-3\ell}{2} - k\ell, \ell\right)$$

$$\equiv 0 \ (\mathrm{mod} \ell^\alpha) \tag{4.13}$$

Observe that $lcm(\ell)\ell^{\alpha-1}\cdot j + k\ell$ and $lcm(\ell)\ell^{\alpha-1}\cdot(j+1) - \frac{\ell^2-3\ell}{2} - k\ell$ are palindromic partners. We rewrite (4.13) by setting $lcm(\ell)\ell^{\alpha-1}j + k\ell = n$ so that $n' = lcm(\ell)\ell^{\alpha-1}(j+1) - \frac{\ell^2-3\ell}{2} - k\ell$. Hence

$$p(n, \ell) - p\left(n', \ell\right) \equiv 0 \pmod{\ell^\alpha} \tag{4.14}$$

and Theorem 3 is proved. The case for $\ell \equiv 3 \pmod 4$ is left to the reader.

$\square$

5. Proof of Corollary 1

To prove Corollary 1 we begin with line (2.4) in Theorem 3. Multiply (2.4) by -1, and set $n = lcm(\ell)\ell^{\alpha-1}g + k\ell - \frac{\ell^2-3\ell}{2}$, giving us the palindromic partner $n' = lcm(\ell)\ell^{\alpha-1}(g+1) - k\ell$ so that we have

$$p\left(lcm(\ell)\ell^{\alpha-1}(g+1) - k\ell, \ell\right) -$$
$$p\left(lcm(\ell)\ell^{\alpha-1}g - \tfrac{\ell^2-3\ell}{2} + k\ell, \ell\right) \equiv 0 \pmod{\ell^\alpha}.$$

Proceed by induction on g. Let $g = 0$ so that for $0 \leq k \leq \frac{\ell-3}{2}$

$$p\left(lcm(\ell)\ell^{\alpha-1} - k\ell, \ell\right) = p\left(lcm(\ell)\ell^{\alpha-1} - k\ell, \ell\right) - 0$$

$$= p\left(lcm(\ell)\ell^{\alpha-1} - k\ell, \ell\right) - p\left(k\ell - \frac{\ell^2-3\ell}{2}, \ell\right) \equiv 0 \pmod{\ell^\alpha}.$$

Now suppose $p\left(lcm(\ell)\ell^{\alpha-1}g - k\ell, \ell\right)$ is true for all $j < g$. Hence,

$$p\left(lcm(\ell)\ell^{\alpha-1}(g+1) - k\ell, \ell\right) -$$
$$p\left(lcm(\ell)\ell^{\alpha-1} + k\ell - \tfrac{\ell^2-3\ell}{2}, \ell\right) \equiv 0 \ (\mathrm{mod} \ell^\alpha)$$

which implies that $p\left(lcm(\ell)\ell^{\alpha-1}j - k\ell, \ell\right) \equiv 0 \pmod{\ell^\alpha}$ for $0 \leq k \leq \frac{\ell-3}{2}$ and $\alpha \geq 1$ by the induction hypothesis. Thus the corollary is proved. $\square$

6. Conclusion and Acknowledgements

There are many more results arising from further extensions of the methods of this paper which are being explored.

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