TORNADO SOLUTIONS FOR SEMILINEAR ELLIPTIC EQUATIONS IN $\mathbb{R}^2$: REGULARITY

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ABSTRACT. We give conditions under which bounded solutions to semilinear elliptic equations $\Delta u = f(u)$ on domains of $\mathbb{R}^2$ are continuous despite a possible infinite singularity of $f(u)$. The conditions do not require a minimization or variational stability property for the solutions. The results are used in a second paper to show regularity for a familiar class of equations.

1. Introduction and Main Results

In this paper we study positive solutions to equations $\Delta u = f(u)$ on domains of $\mathbb{R}^2$ where $f(u)$ is well behaved away from $u = 0$. Our model equation is $\Delta u = u^{-\alpha}$ for $\alpha > 0$. This type of equation has been studied in various papers including [1], [4], and [7]. Positive solutions $u > 0$ will be regular, for example smooth if f is smooth. However, standard regularity estimates depend on $\min u > 0$. Here we give conditions for the existence of interior continuity estimates on solutions. A program for giving precise continuity estimates using the same techniques is a topic of current research. We also consider "generalized solutions" which are limits of smooth positive solutions, and which may have singularities, as in [6]. Regularity properties of such generalized solutions follow from the results for positive solutions. We establish conditions for regularity in terms of the existence of "tornado sequences" of solutions. In the second paper [5], we apply the results presented here to prove regularity properties and existence of singular solutions to $\Delta u = f(u)$ when $f(u) = g(u)u^{-\alpha}$, with $0 \leq C_1 < g(u) < C_2$, g continuous away from $u = 0$.

We begin with the definition of tornado sequences. We will use the notation B_ρ for the ball of radius ρ centered at the origin of $\mathbb{R}^2$.

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Definition 1. *A tornado sequence of solutions to $\Delta u = f(u)$ is given by a number $\epsilon > 0$, a sequence $r_j > 0$, $j = 1, 2, \ldots$, with $r_j \to 0$, and solutions $u_j : B_{r_j} \to \mathbb{R}_+$ such that $u_j > \epsilon$ on ∂B_{r_j} and $\min_{B_{r_j}} u_j \to 0$.*

Given a modulus of continuity $\mu : \mathbb{R}_+ \to \mathbb{R}_+$, i.e. a nondecreasing function with $\mu(\delta) \to 0$ as $\delta \to 0$, we say that a function u on $K \subset \mathbb{R}^n$ is uniformly continuous of class μ if there is a constant C such that $|u(x) - u(y)| \leq C\mu(|x - y|)$ for all x and y in K.

Definition 2. *Let μ be a modulus of continuity. A tornado sequence of class μ is given by a sequence $r_j > 0$ with $r_j \to 0$, constants $C_j \to \infty$, and solutions $u_j : B_{r_j} \to \mathbb{R}_+$ such that $u_j > u_j(0) + C_j\mu(r_j)$ on ∂B_{r_j} and $u_j(0) \to 0$.*

We will assume that Ω is an open domain, and that f satsifies the following for every $\epsilon > 0$:

$$\text{(1)} \qquad \begin{cases} f(u) \leq M(\epsilon) \text{ for } u \geq \epsilon \text{ and} \\ f(u) \text{ is Hölder continuous for } u \geq \epsilon. \end{cases}$$

We may now state the main result. Here, $\tilde{\Omega} \subset\subset \Omega \subset \mathbb{R}^2$ is a compact subset of Ω.

Theorem 1. *Suppose $u_j : \Omega \to \mathbb{R}_+$ is a sequence of solutions to $\Delta u = f(u)$ and $u_j \leq M$. If u_j is not equicontinuous on $\tilde{\Omega}$, then there exists a tornado sequence for $\Delta u = f(u)$.*

In the proof, the tornado sequence is constructed from a subsequence of u_j near points of $\tilde{\Omega}$.

Corollary 1. *If there does not exist a tornado sequence of solutions to $\Delta u = f(u)$, then a bounded family $\mathcal{C}$ of solutions to $\Delta u = f(u)$ on a domain Ω is compact in $C^0(\tilde{\Omega})$ for any $\tilde{\Omega} \subset\subset \Omega$*

Theorem 2. *Suppose $u_j : \Omega \to \mathbb{R}_+$ is a sequence of solutions to $\Delta u = f(u)$ and $u_j \leq M$. Suppose the modulus of continuity μ satisfies $\mu(\delta) > \frac{C}{|\log \delta|^\gamma}$ for some $0 < \gamma < 1$ and for $\delta < \delta_0$. If u_j is not uniformly equicontinuous of class μ on $\tilde{\Omega}$, then there exists a tornado sequence of class μ.*

We note that in contrast to some other results concerning equations of this form, our result does not require stability conditions on the solutions. That is, the equation $\Delta u = f(u)$ is the Euler–Lagrange equation for the functional $\mathcal{F}(u) = \int_\Omega \frac{1}{2}|Du|^2 + F(u)$, where $F(u) = \int_1^u f(t)\,dt$. We do not assume that our solutions u are minimizers of $\mathcal{F}(u)$, or that they are stable in the sense that the second variation

of $\mathcal{F}$ is nonnegative. Minimizers of $\mathcal{F}$ in the set $\{u \geq 0, u \mid_{\partial\Omega} = g\}$ on bounded domains may solve a free boundary problem, which allows for u to be identically zero on a subdomain $\hat{\Omega} \subset \Omega$, not solving the differential equation in $\hat{\Omega}$. This type of problem for the $f(u)$ we consider has been studied for $n \geq 2$ in [3], [4], and [7]. See also the recent book [2] and the references therein. The big questions for these problems deal with the size and regularity of the "free boundary" $\partial\hat{\Omega}$. The regularity of positive stable solutions with $f(u) = Cu^{-\alpha}$, $0 < \alpha \leq 1$, is dealt with in [6].

2. Proofs

We will break up the proof of Theorem 1 into several lemmas below. To begin, we show that equicontinuity can only be violated at points where the functions get close to zero.

Lemma 1. *If the solutions u_j are not equicontinuous on $\tilde{\Omega}$, then there is an $\epsilon > 0$ and a subsequence $u_{j'}$ along with points $x_{j'} \in \tilde{\Omega}$ and $y_{j'} \in \Omega$ with $u_{j'}(x_{j'}) > \epsilon$, $u_{j'}(y_{j'}) \to 0$, and $|x_{j'} - y_{j'}| \to 0$.*

Proof: Equicontinuity is violated if there is an $\epsilon > 0$ and a subsequence u_k with corresponding x_k, y_k in Ω with $|x_k - y_k| \to 0$ and $u_k(x_k) - u_k(y_k) > \epsilon$. Now, assuming the lemma is false, there is no subsequence of u_k with corresponding $\tilde{y}_k$ such that $|x_k - \tilde{y}_k| \to 0$ and $u_k(\tilde{y}_k) \to 0$. That is, there is a $\delta > 0$ so that $u_k > \delta$ on the ball $B_\delta(x_k)$. But then by (1), $|\Delta u_k| \leq M(\delta)$ on $B_\delta(x_k)$ and by the Calderon–Zygmund Inequality and Sobolev Embedding, u_k is continuous, uniformly in k on the ball $B_{\delta/2}(x_k)$, a contradiction.

We now use a classical "log trick" to get a useful estimate of the square integrals of the gradients of u_j.

Lemma 2. *If $0 \leq u \leq M$ is weakly subharmonic on B_{ρ_0}, then there is a $C = C(M)$ such that*

$$\int_{B_\rho} |Du|^2 \leq \frac{C}{|\log \rho|} \qquad \text{for } \rho < \min(\rho_0^2, 1).$$

Proof: A weakly subharmonic function u satisfies the inequality

$$\int_{B_{\rho_0}} Du \cdot D\zeta \leq 0 \qquad \text{for } \zeta \geq 0 \text{ with compact support} \tag{2}$$

We use $\zeta = u\varphi^2$ where φ has support in B_ρ, $\varphi = 1$ on B_{ρ^2}, and $\varphi = \frac{\log r}{\log \rho} - 1$ on $B_\rho \setminus B_{\rho^2}$.

Then we have

$$\begin{aligned}\int |Du|^2\varphi^2 &\leq 2\int u\varphi|Du||D\varphi| \\ &\leq 4\int u^2|D\varphi|^2 \\ &\leq 4M^2\int_{\rho^2}^{\rho}\frac{dr}{r(\log\rho)^2} = \frac{4M^2}{|\log\rho|}.\end{aligned}$$

The lemma follows on replacing ρ by $\sqrt{\rho}$.

We note that this estimate is not sufficient to prove continuity of u directly, as in [10], page 95. See also the remarks in [6].

The following result is akin to the classical Courant–Lebesgue lemma (see [9] and references therein). The original result was used in solving the Plateau Problem for minimal surfaces. We include the proof and a slight improvement we will need for Theorem 2.

Lemma 3. (Courant–Lebesgue) *Suppose that u is C^1 and $h(\rho) = \int_{B_\rho}|Du|^2 \to 0$ as $\rho \to 0$. For any $\delta > 0$ and $0 < \theta < 1$, there is a $\rho_0(\delta,\theta,h) > 0$ such that for all $\rho < \rho_0$, the oscillation*

$$\mathbf{osc}_{\partial B_r} u \leq \delta$$

for all r in a set $A \subset (0,\rho)$ with $|A| \geq \theta\rho$.

Proof: Using the Hölder inequality,

$$\mathbf{osc}_{\partial B_r} u \leq \int_0^{2\pi}\left|\frac{\partial u}{\partial\theta}(r,\theta)\right| d\theta \leq \sqrt{2\pi}\left(\int_0^{2\pi}\left|\frac{\partial u}{\partial\theta}\right| d\theta\right)^{\frac{1}{2}}$$

thus

$$\int_{B_\rho}|Du|^2 \geq \frac{1}{2\pi}\int_0^\rho \frac{1}{r}\left(\mathbf{osc}_{\partial B_r} u\right)^2 dr. \tag{3}$$

Assuming $|A| < \theta\rho$, we then have $\int_{B_\rho}|Du|^2 \geq \frac{\delta^2}{2\pi}|\log\theta|$, a contradiction for ρ small enough depending on δ, θ, and h.

Lemma 4. *Suppose u is C^1 and $h(\rho) = \int_{B_\rho}|Du|^2 \to 0$ as $\rho \to 0$. Consider positive functions $g(\rho)$ and $\eta(\rho)$ both tending to zero as $\rho \to 0$. If for $\rho < \rho_0$*

$$g(\rho) > \sqrt{\frac{2\pi h(\rho)}{|\log(1-\eta(\rho))|}},$$

then for all $\rho < \rho_0$, the oscillation

$$\mathbf{osc}_{\partial B_r} u \leq g(\rho)$$

for all r in a set $A \subset (0,\rho)$ with $|A| \geq \rho(1-\eta(\rho))$.

Proof: From equation (3), we have

$$\frac{1}{2\pi}\int_0^\rho \frac{1}{r}\left(\mathbf{osc}_{\partial B_r} u\right)^2 \, dr \leq h(\rho).$$

Assuming $|A| < \rho(1-\eta(\rho))$, we have

$$\frac{1}{2\pi}\int_{\rho(1-\eta(\rho))}^\rho \frac{1}{r} g(\rho)^2 \, dr \leq h(\rho)$$

and thus

$$g(\rho)^2 |\log(1-\eta(\rho))| \leq 2\pi h(\rho),$$

a contradiction.

We will use the notation Ω_λ for the set of x such that $u(x) > \lambda$. In the following lemmas, we will state the results in arbitrary dimension, although we will only apply them in dimension 2. Here the Sobolev exponent $\kappa = \frac{n}{n-1}$.

Lemma 5. *Suppose $0 < u \leq M$ is subharmonic on $B_{2\rho} \subset \mathbb{R}^n$ with $\rho < 1$ and let $0 < \lambda \leq 1$, $\delta_0 \geq 0$. If $|\Omega_{\delta_0+2\lambda} \cap B_\rho| > 0$, then $|\Omega_{\delta_0+\lambda} \cap B_{2\rho}| \geq C\lambda^{n/2}\rho^n$ for $C = C(M) > 0$.*

The proof of this lemma relies on the following auxiliary result.

Lemma 6. *Suppose $0 < u \leq M$ is subharmonic on B_ρ with $\rho < 1$ and let $0 < \mu \leq 1$, $0 < \sigma < \rho$. Then there is a $C(M) > 0$ with*

$$|\Omega_\lambda \cap B_\rho| \geq C\sigma\sqrt{\mu}|\Omega_{\lambda+\mu} \cap B_{\rho-\sigma}|^{\frac{1}{\kappa}}$$

Proof of Lemma 5: We will iterate the result of Lemma 6, using sequences λ_j and ρ_j.
Set $\lambda_0 = \delta_0 + \lambda$ and $\lambda_j = \delta_0 + 2\lambda - \lambda/2^j = \lambda_{j-1} + \lambda/2^j$.
Set $\rho_0 = 2\rho$ and $\rho_j = \rho + \rho/2^j = \rho_{j-1} - \rho/2^j$.
So, using Lemma 6 with $\mu = \lambda/2^j, \sigma = \rho/2^j$, we get

$$|\Omega_{\lambda_{j-1}} \cap B_{\rho_{j-1}}| \geq C\frac{\rho\lambda^{1/2}}{2^{3j/2}}|\Omega_{\lambda_j} \cap B_{\rho_j}|^{1/\kappa}$$

So,

$$\begin{aligned}
|\Omega_{\lambda_0} \cap B_{\rho_0}| &\geq \frac{C\rho\lambda^{1/2}}{\sqrt{8}} |\Omega_{\lambda_1} \cap B_{\rho_1}|^{\frac{1}{\kappa}} \\
&\geq (C\rho\lambda^{1/2})^{1+\frac{1}{\kappa}} (\sqrt{8})^{-\left(1+\frac{2}{\kappa}\right)} |\Omega_{\lambda_2} \cap B_{\rho_2}|^{\frac{1}{\kappa^2}} \\
&\geq \frac{(C\rho\lambda^{1/2})^{1+\frac{1}{\kappa}+\frac{1}{\kappa^2}+\cdots+\frac{1}{\kappa^{j-1}}}}{(\sqrt{8})^{1+\frac{2}{\kappa}+\frac{3}{\kappa^2}+\cdots+\frac{j}{\kappa^{j-1}}}} |\Omega_{\lambda_j} \cap B_{\rho_j}|^{\frac{1}{\kappa^j}} \\
&\geq (C\rho\lambda^{1/2})^{\sum_0^\infty \kappa^{-j}} (\sqrt{8})^{-\sum_0^\infty (j+1)\kappa^{-j}} |\Omega_{\delta_0+2\lambda} \cap B_\rho|^{\frac{1}{\kappa^j}} \\
&\geq C\lambda^{n/2}\rho^n |\Omega_{\delta_0+2\lambda} \cap B_\rho|^{\frac{1}{\kappa^j}}
\end{aligned}$$

Letting $j \to \infty$ in the last inequality completes the proof.

Proof of Lemma 6: First we fix a nonnegative increasing Lipschitz function $\gamma : \mathbb{R} \to \mathbb{R}$ with $\gamma(t) = 0$ for $t < \lambda$, $\gamma(t) = \mu$ for $t > \lambda + \mu$, $0 \leq \gamma'(t) \leq 1$, and $0 \leq \gamma \leq \mu$. We also choose a cutoff function φ on $\mathbb{R}^n$ with $\varphi = 1$ on $B_{\rho-\sigma}$, $\varphi = 0$ outside B_ρ, and $|D\varphi| \leq 2/\sigma$.
In equation (2), we first substitute $\zeta = u\gamma(u)\varphi$. So,

$$\int \varphi^2 \gamma(u)|Du|^2 + \int u\varphi^2\gamma'(u)|Du|^2 \leq 2\int \varphi u\gamma(u)|Du||D\varphi|$$

Since $\gamma' \geq 0$, we can throw away the second term, and after using Cauchy–Schwartz, we have

$$\int \varphi^2 \gamma(u)|Du|^2 \leq 4\int u^2\gamma(u)|D\varphi|^2. \tag{4}$$

Now we substitute $\zeta = \gamma(u)\varphi^2$. After again using Cauchy–Schwartz, we have

$$\int \varphi^2\gamma'(u)|Du|^2 \leq \int \varphi^2\gamma(u)|Du|^2 + \int \gamma(u)|D\varphi|^2. \tag{5}$$

We will use these inequalities to bound $\int |D(\varphi^2\gamma(u))|$. Note that $|D(\varphi^2\gamma(u))| \leq 2\varphi\gamma(u)|D\varphi| + \varphi^2\gamma'(u)|Du|$. So,

$$\begin{aligned}
\int |D(\varphi^2\gamma(u))| &\leq 2\int \varphi\gamma(u)|D\varphi| + \int \varphi^2\gamma'(u)|Du| \\
&\leq 2\int \varphi\gamma(u)|D\varphi| + \left(\int \varphi^2\gamma'(u)|Du|^2\right)^{\frac{1}{2}} \left(\varphi^2\gamma'(u)\right)^{\frac{1}{2}} \\
&\leq \frac{4\mu}{\sigma}|\Omega_\lambda \cap B_\rho| + \left(\int \varphi^2\gamma(u)|Du|^2 + \int \gamma(u)|D\varphi|^2\right)^{\frac{1}{2}} \left(\varphi^2\gamma'(u)\right)^{\frac{1}{2}} \\
&\leq \frac{4\mu}{\sigma}|\Omega_\lambda \cap B_\rho| + \left(4\int u^2\gamma(u)|D\varphi|^2 + \int \gamma(u)|D\varphi|^2\right)^{\frac{1}{2}} \left(\varphi^2\gamma'(u)\right)^{\frac{1}{2}}
\end{aligned}$$

$$\leq \frac{4\mu}{\sigma}|\Omega_\lambda \cap B_\rho| + \left(\frac{16M^2\mu}{\sigma^2}|\Omega_\lambda \cap B_\rho| + \frac{4\mu}{\sigma^2}|\Omega_\lambda \cap B_\rho|\right)^{\frac{1}{2}} (|\Omega_\lambda \cap B_\rho|)^{\frac{1}{2}}$$

So,

$$\int |D(\varphi^2\gamma(u))| \leq \frac{C(M+1)\mu^{1/2}}{\sigma}|\Omega_\lambda \cap B_\rho| \tag{6}$$

Now we apply the Sobolev Inequality and note that $\varphi^2\gamma(u) = \mu$ on $\Omega_{\lambda+\mu} \cap B_{\rho-\sigma}$:

$$(\mu^\kappa|\Omega_{\lambda+\mu} \cap B_{\rho-\sigma}|)^{\frac{1}{\kappa}} \leq \frac{C\mu^{1/2}}{\sigma}|\Omega_\lambda \cap B_\rho|,$$

which proves the lemma.

We can now prove the main results.

Proof of Theorem 1: Suppose u_j is not equicontinuous on $\tilde{\Omega}$. By Lemma 1, and by translation, there is an $\epsilon > 0$ along with a sequence of solutions u_j on some B_{ρ_0} and a sequence of points $x_j \in B_{\rho_0}$ with $x_j \to 0$, $u_j(x_j) > \epsilon$, and $u_j(0) \to 0$. Thus by Lemma 5 with $\delta_0 = 0$, $\lambda = \epsilon/2$, and $\rho = |x_j|$, there are radii $\rho_j \to 0$ such that u_j is greater than $\epsilon/2$ on a fixed portion of B_{ρ_j}. That is, u satisfies $|\Omega_{\epsilon/2} \cap B_{\rho_j}| \geq \tilde{C}\rho^2$. By Lemmas 2 and 3, with $\delta = \epsilon/4$, $h(\rho) = \frac{C}{|\log\rho|}$, and θ chosen appropriately depending on $\tilde{C}$, there are radii $r_j < \rho_j \to 0$ with $u_j > \epsilon/4$ on ∂B_{r_j}. This completes the proof.

Proof of Theorem 2: As before, supposing u_j is not uniformly μ-continuous on $\tilde{\Omega}$, there is a sequence of solutions u_j on B_{ρ_0} and a sequence of points $x_j \in B_{\rho_0}$ with $x_j \to 0$, $u_j(x_j) > u_j(0) + C_j\mu(|x_j|)$, and $u_j(0) \to 0$. By Lemma 5 with $\delta_0 = u_j(0)$, $\lambda = C_j\mu(|x_j|)/2$, and $\rho = |x_j|$, there are radii $\rho_j \to 0$ such that u_j is greater than $u_j(0)+C_j\mu(\rho_j)/2$ on a portion of $B_{2\rho_j}$ of measure greater than $C_0C_j\mu(\rho_j)\rho_j^2$. That is

$$\left|B_{2\rho_j} \cap \Omega_{u_j(0)+C_j\mu(\rho_j)/2}\right| \geq \tilde{C}_j\mu(\rho_j)\rho_j^2$$

with $\tilde{C}_j \to \infty$. By Lemma 4 with $h(\rho) = C/|\log\rho|$ and $\rho = 2\rho_j$, we have

$$\mathbf{osc}_{\partial B_r} \leq \frac{C}{\sqrt{|\log\rho_j||\log(1-\eta(2\rho_j))|}}$$

on ∂B_r for all r in a set $A \subset (0, 2\rho_j)$ with $|A| \geq 2\rho_j(1-\eta(2\rho_j))$. We choose $\eta(\rho) < \frac{\tilde{C}_j}{8\pi}\mu(\rho/2)$ so that the set

$$B_{2\rho_j} \cap \Omega_{u_j(0)+C_j\mu(\rho_j)/2} \cap \{x : |x| \in A\}$$

is nonempty. Thus there are radii $r_j < 2\rho_j$ so that

$$u_j > u_j(0) + \frac{C_j\mu(\rho_j)}{2} - \frac{C}{\sqrt{|\log\rho_j||\log(1-\mu(\rho_j))|}} \qquad \text{on } \partial B_{r_j}.$$

For $\mu(\delta) \geq |\log\delta|^{-1/3}$, this implies

$$u_j > u_j(0) + \frac{C_j\mu(\rho_j)}{4} \qquad \text{on } \partial B_{r_j},$$

for sufficiently large j. Thus u_j forms a tornado sequence of class μ. At this point we have proven the result for all $\gamma \leq \gamma_0 = \frac{1}{3}$.
Now suppose $1/3 \leq \gamma_k < 1$ and the result has been proven for $\gamma \leq \gamma_k$. Let $\gamma > \gamma_k$ and let u_j be a sequence of solutions not equicontinuous of class $\mu(\delta) = |\log\delta|^{-\gamma}$. If we assume, for a contradiction, that there is no tornado sequence of class μ, then by the assumption, all bounded solutions are equicontinuous of class $\nu_k(\delta) = |\log\delta|^{-\gamma_k}$. Thus, for any ρ, a solution $u \leq M$ satisfies $|u(x) - \min_{B_\rho} u| \leq C|\log\rho|^{-\gamma_k}$ for $x \in B_\rho$. So, using the test function $(u - \min_{B_\rho} u)\varphi^2$ as in Lemma 2, we have

$$\int_{B_\rho} |Du|^2 \leq \frac{C}{|\log\rho|^{1+2\gamma_k}}.$$

The above argument using Lemma 4 then shows that for any modulus ν with $\nu(\delta) \geq \nu_{k+1}(\delta) = |\log\delta|^{-\gamma_{k+1}}$, where $\gamma_{k+1} = \frac{1+2\gamma_k}{3}$, u_j are equicontinuous of class ν unless there is a tornado sequence of class ν. So, we have proven the result for $\gamma \leq \gamma_{k+1}$. We may continue the bootstrap sequence ν_k until we reach a contradiction with $\mu < \nu_k$.

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