MATH 391

Putnam Seminar Fall 2015

Cognitive Control:
What am I doing? Why am I doing it? How will it help me?
Problem-solve away from frustration. Don't give up.
Heuristics: Search for a pattern. Draw a figure. Choose effective notation. Work backward.
Prior Problems:

1. Assume that cucumbers are typically 99% water by weight. Suppose that 500 pounds of cucumbers are allowed to sit in the sun for an entire day. The following morning, what remains after evaporation is 98% water. What then is the weight of what remains?

2. St. Mary's has an elimination tennis tournament with $1,025$ tennis players. In each round the players pair up at random to play a match, with possibly one person not playing if the number of players is odd. The winners of each round advance to the next along with the possible odd player out. In the end there is one champion. What is the total number of matches that are played altogether in all rounds of the tournament?

3. Show that a number is divisible by 9 if and only if the sum of its digits is divisible by 9.

4. Let A be the sum of the digits of the number 4444^{4444} and let B be the sum of the digits of A. What is the sum of the digits of B?

5. A particle moving on a straight line starts from rest and attains a velocity v_0 after traversing a distance s_0. If the motion is such that the acceleration was never increasing, find the maximum time for the traverse.

6. If n is a positive integer such that $2n+1$ is a perfect square, show that $n+1$ is the sum of two successive perfect squares.

7. Show that, for any positive integer n, $1^n + 2^n + 3^n + 4^n$ is divisible by 5 if an only if n is not divisible by 4.

8. Find all differentiable functions $f : (0,\infty) \to (0,\infty)$ for which there is a positive real number a such that

$$f'\left(\frac{a}{x}\right) = \frac{x}{f(x)} \qquad \text{for all } x > 0.$$

9. In a round-robin basketball tournament with n teams $T_1, T_2, \ldots, T_n$, where $n > 1$, each team plays one game with each of the other teams and no ties can occur. Let W_r and L_r be the number of games won and lost, respectively, by team T_r. Show that

$$\sum_{r=1}^{n} W_r^2 = \sum_{r=1}^{n} L_r^2.$$

The Pigeonhole Principle: If $n+1$ or more objects are placed in n boxes, then at least one box contains more than one object.

More generally we can say

If $kn+1$ or more objects are placed in n boxes, than at least one box contains more than k objects.

Examples:

- If a coin is tossed three times, two of the tosses have the same result.
- In a 27 word english sentence, at least two words start with the same letter.
- In a class of 102 students taking an exam to be graded out of 100 points, at least two students get the same grade.
- How many people need to be in a room to guarantee that two of them have the same birthday?

Plan of Attack:

1. Decide what the pigeons are. These are the things you want more than one of to have some property.
2. Make the pigeonholes. There are two things to make sure of here. First, if there are two or more pigeons in the same pigeonhole, they have the property you want. Second, there should be fewer pigeonholes than pigeons.
3. Make a rule for assigning pigeons to pigeonholes. Sometimes it is necessary to make the right rule here in order to get the property you want.

Modular Arithmetic

The expression $a \equiv b \pmod{n}$, pronounced "a is congruent to b modulo n", means that $a - b$ is an integer multiple of n. For instance, $(-43) - 37 = -80$ so that $-43 \equiv 37 \pmod{4}$. Given a, there is only one value b between 0 and $n-1$ so that $a \equiv b \pmod{n}$. We call b the residue of a modulo n and write $b = (a \bmod n)$. In fact, for positive a, b is just the remainder when dividing a by n.
Quick Facts:

- A number and ints negative are usually not congruent: $2 \not\equiv -2 \pmod{9}$, since $2-(-2) = 4$ is not a multiple of 9. This is a source of many mistakes.
- Suppose that $a \equiv b$ and $c \equiv d \pmod{n}$. Then $a + c \equiv b + d \pmod{n}$ and $a \cdot c \equiv b \cdot d \pmod{n}$.
- Dividing is not so simple: $6 \equiv 36 \pmod{1}0$, but dividing by 2 would give $3 \equiv 18 \pmod{1}0)$, which is not true!

  The problem above is that 2 divides 10 (think about it). We can do two things:

  - Divide by a number k relatively prime to n: $6 \equiv 36 \pmod{10}$, so dividing by 3 gives $2 \equiv 12 \pmod{10}$.
  - Divide all three numbers by a number k which is a divisor of n: $6 \equiv 36 \pmod{10}$ so dividing by 2 gives $3 \equiv 18 \pmod{5}$.
- You can also reduce n alone: $7 \equiv 13 \pmod{6} \longrightarrow 7 \equiv 13 \pmod{3}$. But this does not work in the opposite direction: $13 \equiv 16 \pmod{3}$, but $13 \not\equiv 16 \pmod{6}$.
- To compute exponents we use Euler?s Theorem:

  $$\text{If } a \text{ is relatively prime to } n, \text{ then } a^{\varphi(n)} \equiv 1 \pmod{n}.$$

  (Here, $\varphi(a)$ is the number of integers between 1 and n, relatively prime to n.)
- A useful result concerning factorials is Wilson?s Theorem:

  $$\text{The number } p \text{ is prime if and only if } (p-1) \equiv -1 \pmod{p}.$$

New Problems:

1. Try these pigeon problems:
   (a) Given seven distinct numbers between 1 and 11, show that some pair of them sum to 12.
   (b) Over a 30 day period, Amy walked the dog at least once a day, and a total of 45 times in all. Show that there was a period of consecutive days during which she walked the dog exactly 14 times.
   (c) Show that $(a-b)(b-c)(a-c)$ is always an even integer if a, b, and c are integers.
   (d) What is the maximum number of rooks that you can put on an 8×8 chess-board so that no two rooks can hit each other?
2. Make safe the modulus:
   (a) Show that n is congruent to the sum of the digits of $n \pmod 9$.
   (b) Show that n is congruent to the sum of the digits of $n \pmod 3$.
   (c) How do you find $n \pmod{10}$ and $n \pmod 5$? What about $n \pmod{11}$?
3. Show that, given a 7-digit number, you can cross out some digits at the beginning and at the end such that the remaining number is divisible by 7. For example, you can cross out the first 3 and the last 2 digits of 1294961 to get 49.
4. Show that some power of 7 ends in "0000001."
5. What is the remainder when dividing $1 + 2 + \cdots + 100$ by 15?
6. Show that some positive multiple of 21 has 241 as its final three digits.
7. Show that if n divides a single Fibonacci number, then it will divide infinitely many Fibonacci numbers.
8. How many distinct divisors does the number 2015 have?
9. Show that every positive integer is a sum of one or more numbers of the form $2^r 3^s$, where r and s are nonnegative integers and no summand divides another. (For example, $23 = 9 + 8 + 6$.)